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BP TORSION IN FINITE *H*-SPACES

BY

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ABSTRACT. Let p be odd and (X, μ) a 1-connected mod p finite H-space. It is shown that for $n > 1$ the Morava K-theories, $k(n)_*(X)$ and $k(n)^*(X)$, have no higher v_n torsion. Also examples are constructed to show that v_1 torsion in $BP\langle 1\rangle^*(X)$ can be of arbitrarily high order.

1. Introduction. Let p be a prime. Let $\mathbf{Z}_{(p)}$ be the integers localized at the prime p and let $\mathbf{Z}/p$ be the integers reduced mod p. By an H-space (X, μ) we will mean a pointed topological space X which has the homotopy type of a connected CW complex of finite type together with a basepoint preserving map $\mu : X \times X \to X$ with two-sided homotopy unit. By a (mod p) finite H-space (X, μ), we will mean an H-space such that $H^*(X; \mathbf{Z}/p)$ is a finite-dimensional $\mathbf{Z}/p$ module. If (X, μ) is a 1-connected (mod p) finite H-space, then $H_*(X) = H_*(X; \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$ has no higher p torsion for p odd (see [**19**]) and a similar result is conjectured to be true for $p = 2$ (see [**13**]). By no higher p torsion we mean that $px = 0$ for all $x \in$ Torsion $H_*(X)$.

Let $BP_*(X)$ be the Brown Peterson homology of X (see [**2**] and [**25**]). Then $BP_*(X)$ is a module over $BP_*(\text{pt}) = \mathbf{Z}_{(p)}[v_1, v_2, \ldots]$ (dim $v_s = 2p^s - 2$). Hence, besides p torsion, we can speak of v_s torsion in $BP_*(X)$ for $s \geq 1$. Based on the above result for ordinary homology, we ask the following obvious question. For each $s \geq 0$ (let $v_0 = p$) can there be higher v_s torsion in $BP_*(X)$ when (X, μ) is a 1-connected (mod p) finite H-space? That is, is it necessary true that, for any $x \in BP_*(X)$ and any $s \geq 0$, $v_s^n x = 0$ for $n > 1$ always implies $v_s x = 0$. In general the answer is negative (see §8). However, if we pass to certain theories associated with BP, then restrictions on torsion can be obtained.

For each $n \geq 1$ let $k(n)_*(X)$ be the connected Morava K-theory of X (see [**12**]). It is a module over $k(n)_*(\text{pt}) = \mathbf{Z}/p[v_n]$ (dim $v_s = 2p^s - 2$). There is a canonical ring homomorphism $BP_*(X) \to k(n)_*(X)$ which maps v_n to v_n and v_i to 0 for $i \neq n$. The only type of torsion which can occur in $k(n)_*(X)$ is v_n torsion. We will show

THEOREM 1:1. *Let p be odd and let (X, μ) be a* 1-*connected* (mod p) *finite H-space. Then $k(n)_*(X)$ has no higher v_n torsion. That is, $v_n x = 0$ for all $x \in$* Torsion $k(n)_*(X)$.

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If we let $v_0 = p$ and interpret $k(0)_*(X)$ as $H_*(X)$, then 1:1 is an obvious analogue to the torsion result for $H_*(X)$. Our proof of 1:1 is also analogous to that used to prove lack of higher torsion in $H_*(X)$. We employ a Bockstein spectral sequence argument. For each $n \geqslant 1$ we have an exact couple.

$$\begin{array}{ccc} k(n)_*(X) & \xrightarrow{v_n} & k(n)_*(X) \\ & \Delta_n \nwarrow \quad \swarrow \rho_n & \\ & H_*(X; \mathbf{Z}/p) & \end{array} \qquad (T_n)$$

where v_n denotes multiplication by v_n and ρ_n is the canonical map from $k(n)_*(X)$ to $H_*(X; \mathbf{Z}/p)$ ("reduction mod v_n"). This exact couple induces a Bockstein spectral sequence $\{B^r\}$ and we prove 1:1 by showing that $B^2 = B^\infty$ (see §3). The differential d^1 acting on $B^1 = H_*(X; \mathbf{Z}/p)$ can be identified with the Milnor cohomology operation Q_n. Thus our proof of 1:1 also yields the following lifting result (indeed it is equivalent to 1:1).

THEOREM 1:2. *Let p be odd and let* (X, μ) *be a* 1-*connected* mod p *finite H-space. Then* $x \in H_*(X; \mathbf{Z}/p)$ *lies in the image of* $\rho_n : k(n)_*(X) \to H_*(X; \mathbf{Z}/p)$ *if and only if* $Q_n(x) = 0$.

These results can be converted from homology to cohomology. This follows from the fact that, for a finite H-space, X^+ ($= X$ plus a disjoint basepoint) is its own S-dual (see [6]). Thus there are natural isomorphisms $\tilde{h}^*(X) = \tilde{h}_*(X)$ for $h = H\mathbf{Z}/p$ and $k(n)$ (see [1]). In particular, the following statements are dual to 1:1 and 1:2.

THEOREM 1:3. *Let p be odd and let* (X, μ) *be a* 1-*connected* (mod p) *finite H-space. Then* $k(n)^*(X)$ *has no higher* v_n *torsion.*

THEOREM 1:4. *Let p be odd and let* (X, μ) *be a* 1-*connected* (mod p) *finite H-space. Then* $x \in H^*(X; \mathbf{Z}/p)$ *lies in the image of* $\rho_n : k(n)^*(X) \to H^*(X; \mathbf{Z}/p)$ *if and only if* $Q_n(x) = 0$.

The proofs of Theorems 1:1 and 1:2 occupy §§2–6. Let $K(n)_*(X)$ be the Morava K-theory of X. It is a module over $K(n)_*(\mathrm{pt}) = \mathbf{Z}/p[v_n, v_n^{-1}]$ and can be obtained from $k(n)_*(X)$ by localizing with respect to v_n. We prove 1:1 and 1:2 by studying $K(n)_*(X)$. We begin by studying $H^*(X; \mathbf{Z}/p)$ as a Q_n differential Hopf algebra (see §2). We then use this information to calculate $K(n)_*(X)$ as a $\mathbf{Z}/p[v_n, v_n^{-1}]$ module in two distinct ways. First of all, we dualize our information for $H^*(X; \mathbf{Z}/p)$ and calculate B^2 = the homology of $H_*(X; \mathbf{Z}/p)$ ($= B^1$) with respect to the differential Q_n. Letting $N = \operatorname{rank} B^2$ (as a $\mathbf{Z}/p$ module) we have the inequality $N \geqslant \operatorname{rank} K(n)_*(X)$ as a $\mathbf{Z}/p[v_n, v_n^{-1}]$ module with equality if and only if $B^2 = B^\infty$ (see §3). Secondly, we use our information for $H^*(X; \mathbf{Z}/p)$ to calculate the algebra $H_*(\Omega X; \mathbf{Z}/p)$ via an Eilenberg-Moore spectral sequence (see §4). We then calculate the algebra $K(n)_*(\Omega X)$ (see §5) and use this information to deduce, via another Eilenberg-Moore type spectral sequence, that $\operatorname{rank} K(n)_*(X) \geqslant N$ (see §6). Our two approaches now yield that $B^2 = B^\infty$.

There are many potential extensions of the above results. Most are not possible. First of all Theorems 1:1–1:4 do not extend to the case $p = 2$. The work of Hodgkins in III of **[10]** shows that the exceptional Lie groups $X = E_7$ and E_8 have higher v_1 torsion in $k(1)^*(X)$ when $p = 2$. However Theorems 1:1–1:4 may be true if we restrict our attention to 1-connected (mod 2) finite H-spaces such that $H^*(X; \mathbf{Z}/2)$ is primitively generated. Results similar to those obtained in §2 are likely. But it is not known if the multiplicative properties of $k(n)$ and $K(n)$ used in §§5 and 6 are true for $p = 2$. Thus we should also point out that different proofs of 1:1 and 1:2 may be possible. If the type of implication theorems proven by Browder for the classical Bockstein spectral sequence (see **[3]**) have analogues for the spectral sequences used in this paper then $B^2 = B^\infty$ is a simple consequence of the results of §2.

Next, Theorems 1:1 and 1:3 do not extend to *BP* theories in which more than one type of torsion is allowable. In §§7–9 we will study the case of $BP\langle 1\rangle$ theory. It seems typical of the general situation. Recall that $BP\langle 1\rangle^*(\text{pt}) = \mathbf{Z}_{(p)}[v_1]$ (see **[11]**). For $p = 2$ let X be the exceptional Lie group G_2. For p odd let X be the finite H-space constructed by Harper in **[8]**. We will prove

THEOREM 1:5. *For any integer* $m \geq 1$, $BP\langle 1\rangle^*(\prod_{i=1}^m X)$ *contains an element* x *satisfying* $v_1^{m-1}x \neq 0$ *while* $v_1^m x = 0$.

On the other hand generalizations of 1:2 and 1:4 do seem possible. One possible generalization would be that for mod odd 1-connected finite H-spaces $x \in H^*(X; \mathbf{Z}/p)$ lies in the image of the Thom map $BP^*(X) \to H^*(X; \mathbf{Z}/p)$ if and only if $Q_0(x) = Q_1(x) = \cdots = 0$. This result has been verified for the simple, simply-connected, compact Lie groups by Yagita (see **[27]** and **[28]**).

Throughout this paper we will always assume, unless otherwise indicated, that p *is an odd prime and that* (X, μ) *is a* 1-*connected* (mod p) *finite H-space*.

We close this section with some remarks on the Steenrod algebra and on Hopf algebras. Regarding the Steenrod algebra $A^*(p)$, we will rely on Milnor's treatment from **[22]**. Besides the usual left action of $A^*(p)$ on $H^*(X; \mathbf{Z}/p)$, we will also use the left action of $A^*(p)$ on $H_*(X; \mathbf{Z}/p)$ obtained by duality. That is $\langle\chi(\theta)a, x\rangle = (-1)^{|\theta|\,|a|}\langle a, \theta x\rangle$ for any $\theta \in A^*(p)$, $a \in H_*(X; \mathbf{Z}/p)$ and $x \in H^*(X; \mathbf{Z}/p)$. ($\chi : A^*(p) \to A^*(p)$ is the canonical antiautomorphism.) It is with respect to this action that we have the already mentioned identity $d^1 = Q_n$ and $B^1 = H_*(X; \mathbf{Z}/p)$ for the Bockstein spectral sequence arising from T_n. As for the action of $A^*(p)$ on $H^*(X; \mathbf{Z}/p)$, observe that it is an unstable action. That is, $\mathcal{P}^n$ acts trivially in dimensions $< 2n$ while $\mathcal{P}^n(x) = x^p$ for all x in dimension $2n$.

Regarding Hopf algebras, the general reference is **[23]**. We will deal with Hopf algebras over both $\mathbf{Z}/p$ and $\mathbf{Z}/p[v_n, v_n^{-1}]$. Given an H-space (X, μ) then $H^*(X; \mathbf{Z}/p)$ and $H_*(X; \mathbf{Z}/p)$ have natural structures as $\mathbf{Z}/p$ Hopf algebras over the Steenrod algebra. The structures are induced by μ and the diagonal map $\Delta : X \to X \times X$ and are dual to one another. Similarly $K(n)^*(X)$ and $K(n)_*(X)$ are dual $\mathbf{Z}/p[v_n, v_n^{-1}]$ Hopf algebras (see **[26]** for the properties of $K(n)$). Given a Hopf algebra A, we will use $P(A)$ and $Q(A)$ to denote primitives and indecomposables, respectively. If A and A^* are dual Hopf algebras, then $Q(A)$ and $P(A^*)$ are dual in the sense of a quotient module of A being dual to a submodule of A^*. If A is a Hopf algebra over $A^*(p)$, then both $P(A)$ and $Q(A)$ inherit $A^*(p)$ structures from $H^*(X; \mathbf{Z}/p)$. Furthermore, the duality between $Q(A)$ and $P(A^*)$ is in the sense of $A^*(p)$ modules. If A is a commutative, associative Hopf algebra over $\mathbf{Z}/p$ then A is isomorphic, as an algebra, to a tensor product $\bigotimes A_i$ of Hopf algebras over $\mathbf{Z}/p$ where each A_i is generated as an algebra by a single element a_i. Such a decomposition is called a Borel decomposition of A and the elements $\{a_i\}$ are called generators of the decomposition. An element $x \in A$ is said to be of height n if $x^{n-1} \neq 0$ while $x^n = 0$. The height of a Borel generator a_i is 2 if a_i has odd degree and a power of p or ∞ if a_i has even degree.

Unless otherwise indicated we will work in the category of graded $\mathbf{Z}/p$ modules. In particular tensor products will be over $\mathbf{Z}/p$ unless otherwise indicated. Given a graded set S we will use $E(S)$, $\mathbf{Z}/p[S]$ and $\Gamma(S)$ to denote the graded exterior, polynomial and divided polynomial Hopf algebras over $\mathbf{Z}/p$ generated by S. We will use $T(S)$ to denote the polynomial Hopf algebra with all elements truncated at height p.

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2. The structure of $H^*(X; \mathbf{Z}/p)$. In this section we will discuss the Hopf algebra structure of $H^*(X; \mathbf{Z}/p)$. Our results are extensions of those obtained in [19]. We begin with the Steenrod module structure of $Q = Q(H^*(X; \mathbf{Z}/p))$.

$$Q^{\text{even}} = \sum_{n > 1} \beta_p \mathcal{P}^n Q^{2n+1}. \tag{2:1}$$

(2:2) If $Q^{2n} \neq 0$ then $n = p^k + \cdots + p + 1$ or $n = p^{k+1} + \cdots + p^{l+1} + p^{l-1} + \cdots + p + 1$ $(k \geqslant l \geqslant 1)$.

In particular $Q^{2n} \neq 0$ implies n has a p-adic expansion of the form $n = p^{s_1} + \cdots + p^{s_k}$ $(s_1 > s_2 > \cdots > s_k)$. Such an integer is said to be binary (with respect to p). We will now show that the Steenrod module structure of Q^{even} can be read off the p-adic expansion of the various n such that $Q^{2n} \neq 0$. Given $n = \sum n_s p^s$ let $\omega(n) = \sum n_s$. In particular, if n is binary then $\omega(n)$ is the number of nonzero terms in its p-adic expansion.

(2:3) The function ω defines a splitting of Q^{even} as a Steenrod module. That is, $Q^{\text{even}} = \bigoplus_{s \geqslant 2} M_s$ where $M_s = \sum_{\omega(n)=s} Q^{2n}$.

(2:4) Given $n = p^{k+1} + \cdots + p^{l+1} + p^{l-1} + \cdots + p + 1$ let

$$m = \begin{cases} p^{k+1} + \cdots + p^{l+2} + p^l + \cdots + p + 1 & \text{if } k > l, \\ p^k + \cdots + p + 1 & \text{if } k = l. \end{cases}$$

Then $Q^{2n} = \mathcal{P}^{p^l} Q^{2m}$.

To visualize these structure theorems write down the sequence

$$(p+1, p^2+1)(p^2+p+1, p^3+p+1, p^3+p^2+1)(p^3+p^2+p+1, \ldots)$$

representing the possible n for which $Q^{2n} \neq 0$. Each set of brackets contains the dimensions occurring in a particular M_s. Thus no Steenrod power acts across any bracket. On the other hand, any two adjacent dimensions within a given set of brackets are connected by the appropriate Steenrod power. For example

$$p^2 + p + 1 \xrightarrow{\mathcal{P}^{p^2}} p^3 + p + 1 \xrightarrow{\mathcal{P}^{p}} p^3 + p^2 + 1.$$

These results on Q can be used to obtain theorems about the algebra structure of $H^*(X; \mathbf{Z}/p)$. Since $H^*(X; \mathbf{Z}/p)$ is commutative and associative it has a Borel decomposition $\bigotimes A_i$ with generators $\{a_i\}$. Since the Borel generators project to a $\mathbf{Z}/p$ basis of Q, it follows that 2:2 restricts the dimensions in which generators can lie. Also, since the pth power of any $x \in H^*(X; \mathbf{Z}/p)$ can be identified with a Steenrod power operating on x, 2:3 and 2:4 can be used to restrict the height of each generator.

$$\begin{aligned}&\text{If } |a_i| = 2(p^k + \cdots + p + 1) \text{ then } a_i^{p^2} = 0.\\&\text{If } |a_i| = 2(p^{k+1} + \cdots + p^{l+1} + p^{l-1} + \cdots + p + 1) \text{ then}\\&a_i^p = 0.\end{aligned} \tag{2:5}$$

For all of the above results, see [19]. We will now use these results to study how the Milnor elements $\{Q_s\}_{s \geq 0}$ in $A^*(p)$ act on $H^*(X; \mathbf{Z}/p)$. Our basic result is

THEOREM 2:6. *$Q^{2n} = Q_s Q^{2n-2p^s+1}$ if $n \geq p^s$.*

Most of the remaining part of this section will be spent in proving 2:6. We first observe that the operations $\{Q_s\}_{s \geq 0}$ satisfy the relations:

$$Q_s Q_t = -Q_t Q_s \quad \text{for } s, t \geq 0, \tag{2:7}$$

$$Q_s \mathcal{P}^t = \mathcal{P}^t Q_s - Q_{s+1} \mathcal{P}^{t-p^s} \quad \text{for } s, t \geq 0 \tag{2:8}$$

(here, as elsewhere, we will assume $\mathcal{P}^i = 0$ for $i < 0$). By repeated applications of 2:8 we obtain the relations

$$Q_s \mathcal{P}^t = \sum_{i \geq 0} (-1)^i \mathcal{P}^{t-(p^s + \cdots + p^{s+i-1})} Q_{s+i} \quad \text{for } s, t \geq 0. \tag{2:9}$$

We will prove 2:6 by induction on n, so we can make the following assumption:

INDUCTION HYPOTHESIS. Theorem 2:6 is true in dimensions $< 2n$.

We break our proof of 2:6 for $2n$ into a number of cases.

Case I. $n = p^k + \cdots + p + 1$. By an inductive argument on s we can prove that for $0 \leq s \leq k$ we have the relation

$$Q^{2(p^k + \cdots + p+1)} = Q_s \mathcal{P}^{p^{k-1} + \cdots + p^s} Q^{2(p^{k-1} + \cdots + p+1)+1}. \tag{$*$}$$

The initial case is 2:1 since $Q_0 = \beta_p$. Assuming s we deduce $s + 1$ by using 2:8. (The term $\mathcal{P}^{p^{k-1} + \cdots + p^s} Q_s$ disappears by 2:2.)

Case II. $n = p^{k+1} + \cdots + p^{l+1} + p^{l-1} + \cdots + p + 1$ *and* $s \leq l - 1$. Let m be as in 2:4. Thus $Q^{2n} = \mathcal{P}^{p^l} Q^{2m}$. Also, by the induction hypothesis, $Q^{2m} = Q_s Q^{2m-2p^s+1}$. Therefore

$$Q^{2n} = \mathcal{P}^{p^l} Q_s Q^{2m-2p^s+1} = Q_s \mathcal{P}^{p^l} Q^{2m-2p^s+1}. \tag{$**$}$$

The second equality follows from 2:9 (the terms $\mathcal{P}^{p^l - (p^s + \cdots + p^{s+i-1})} Q_{s+i}$ disappear by 2:2).

Case III. $n = p^{k+1} + \cdots + p^{l+1} + p^{l-1} + \cdots + p + 1$ *and* $s \geq l + 1$. Since we must have $n \geq p^s$ we can actually assume that $l + 1 \leq s \leq k + 1$. Again let m be as in 2:4. Thus $Q^{2n} = \mathcal{P}^{p^l} Q^{2m}$. First of all, suppose $k > l$. Then $m \geq p^{k+1} \geq p^s$. Hence, by the induction hypothesis, $Q^{2m} = Q_s Q^{2m-2p^s+1}$. Then

$$Q^{2n} = \mathcal{P}^{p^l} Q_s Q^{2m-2p^s+1} = Q_s \mathcal{P}^{p^l} Q^{2m-2p^s+1}. \tag{***}$$

The second equality follows from 2:8 (the term $Q_{s+1}\mathcal{P}^{p^l-p^s}$ disappears since $p^l - p^s < 0$). Secondly, suppose $k = l$. Then $s = l + 1$ and $m = p^l + \cdots + p + 1$. By the induction hypothesis $Q^{2m} = Q_l Q^{2m-2p^l+1}$. Thus

$$Q^{2n} = \mathcal{P}^{p^l} Q_l Q^{2m-2p^l+1} = Q_{l+1} Q^{2m-2p^l+1}. \tag{****}$$

The second equality follows from 2:8 (the term $Q_l \mathcal{P}^{p^l}$ disappears since $\mathcal{P}^{p^l}$ acts trivially in dimensions $< 2p^l$).

Case IV. $n = p^{k+1} + \cdots + p^{l+1} + p^{l-1} + \cdots + p + 1$ *and* $s = l$. We want to show $Q^{2n} = Q_l Q^{2n-2p^l+1}$. We will deduce our result by using secondary operations. (Unstable) secondary operations are associated to (unstable) Adem relations. In dimensions $\leq 2r + 2$ we have the unstable relation

$$Q_0 \mathcal{P}^{r+p^{l-1}} Q_{l-1} = -Q_l Q_0 \mathcal{P}^r. \tag{2:10}$$

Relation 2:10 is a simple consequence of 2:7 and 2:8 (see §3 of **[13]**). This unstable relation gives rise to an unstable "secondary operation" ϕ defined on Q (see **[18]**). Let $B(q)$ be the sub-Hopf algebra of $H^*(X; \mathbf{Z}/p)$ generated over $A^*(p)$ by $\sum_{i \leq q} H^i(X; \mathbf{Z}/p)$. This filtration $\{B(q)\}$ of $H^*(X; \mathbf{Z}/p)$ defines a filtration $\{F_q Q\}$ of Q. Our secondary operation ϕ is used to prove the following:

LEMMA 2:11. *Given* $\bar{x} \in Q^{2r+1}$ *and* $\bar{y} = Q_{l-1}(\bar{x}) \in Q^{2r+2p^{l-1}}$, *pick* q *such that* $\bar{y} \in F_{q+1}Q$, $\bar{y} \notin F_q Q$. *Then pick* $a \in P_{2r+2p^{l-1}}(H_*(X; \mathbf{Z}/p))$ *such that* $\langle a, \bar{y} \rangle \neq 0$ *and* $\langle a, B(q) \rangle = 0$ (*such choices are always possible*). *If we can find a representative* $x \in H^{2r+1}(X; \mathbf{Z}/p)$ *for* $\bar{x}$ *such that* $Q_0 \mathcal{P}^r(x) \in B(q) \cdot B(q)$ *then* $Q_l(a) \neq 0$.

This lemma is an application of Theorem 3:1:1 of **[19]** to the Adem relation 2:10. Actually 3:1:1 of **[19]** allows a second possible conclusion, namely $a^p \neq 0$. However Theorem 5:4:1 of **[19]** eliminates this possibility.

Recall that $n = p^{k+1} + \cdots + p^{l+1} + p^{l-1} + \cdots + p + 1$. We will apply Lemma 2:11 with $r = n - p^{l-1}$ and $q = 2(p^{k-1} + \cdots + p + 1)$. To apply the lemma pick an arbitrary $0 \neq \bar{y} \in Q^{2n}$.

(i) $\bar{y} \in F_{q+1}Q$. It follows from the relations in (*) and (**) for $s = l - 1$ that we can trace $\bar{y}$ back to an element $\bar{z} \in Q^{2(p^{k-1} + \cdots + p + 1) + 1}$. More exactly we have the identities

$$\begin{aligned} Q^{2n} &= \mathcal{P}^{p^l} \mathcal{P}^{p^{l+1}} \cdots \mathcal{P}^{p^k} Q_{l-1} \mathcal{P}^{p^{k-1} + \cdots + p^{l-1}} Q^{2(p^{k-1} + \cdots + p+1)+1} \\ &= Q_{l-1} \mathcal{P}^{p^l} \mathcal{P}^{p^{l+1}} \cdots \mathcal{P}^{p^k} \mathcal{P}^{p^{k-1} + \cdots + p^{l-1}} Q^{2(p^{k-1} + \cdots + p+1)+1}. \end{aligned}$$

This concludes the proof of (i).

(ii) $\bar{y} \notin F_q Q$. Every element of $A^*(p)$ is a polynomial in the elements $\{\mathcal{P}^s\}_{s \geq 1}$ and $\{Q_s\}_{s \geq 0}$. Indeed, by 2:9, every element of $A^*(p)$ can be written in terms of the monomials $\{\mathcal{P}^{n_1} \cdots \mathcal{P}^{n_i} Q_{m_1} \cdots Q_{m_j}\}$. Thus, if $\bar{y}$ can be traced back to an element in dimension $\leqslant 2(p^{k-1} + \cdots + p + 1)$ then it can be done using the operations $\mathcal{P}^{n_1} \cdots \mathcal{P}^{n_i} Q_{m_1} \cdots Q_{m_j}$. Now $\bar{y} \in M_k$. By 2:3 any nonzero element of M_k can only be traced back to other elements in M_k via the operations $\mathcal{P}^{n_1} \cdots \mathcal{P}^{n_i}$. On the other hand nonzero elements of M_k cannot be traced back to dimensions $\leqslant 2(p^{k-1} + \cdots + p + 1)$ using the operations $Q_{m_1} \cdots Q_{m_j}$. For, by **[3]**, $Q_s Q_t = 0$ on Q for any $s, t \geqslant 0$. Thus we need only consider the operations $\{Q_s\}_{s > 0}$. And, considering the dimensions in which nontrivial elements of M_k lie (see 2:2), and subtracting the dimension of Q_s $(= 2p^s - 1)$, we always end up with integers < 0 or $\geqslant 2(p^{k-1} + \cdots + p + 1) + 1$. This concludes the proof of (ii).

Using the relations in (i) we can pick x and $\bar{x}$. Pick $\bar{z} \in Q^{2(p^{k-1} + \cdots + p + 1) + 1}$ such that

$$\bar{y} = Q_{l-1} \mathcal{P}^{p^l} \cdots \mathcal{P}^{p^k} \mathcal{P}^{p^{k-1} + \cdots + p^{l-1}}(z).$$

Let $z \in H^{2(p^{k-1} + \cdots + p + 1) + 1}(X; \mathbf{Z}/p)$ be any representative for $\bar{z}$. Let

$$x = \mathcal{P}^{p^l} \cdots \mathcal{P}^{p^k} \mathcal{P}^{p^{k-1} + \cdots + p^{l-1}}(z) \quad \text{and} \quad \bar{x} = \mathcal{P}^{p^k} \cdots \mathcal{P}^{p^l} \mathcal{P}^{p^{k-1} + \cdots + p^{l-1}}(\bar{z}).$$

By definition

(iii) $\bar{y} = Q_{l-1}(\bar{x})$.

Furthermore

(iv) $Q_0 \mathcal{P}^r(x) \in B(q) \cdot B(q)$.

By 2:1 and 2:2 $Q_0 \mathcal{P}^r(\bar{x}) = 0$ in Q, that is $Q_0 \mathcal{P}^r(x)$ is decomposable. Also, since $u^*(z) = z \otimes 1 + 1 \otimes z + z'$ where $z' \in B(q) \otimes B(q)$, it follows that $u^*(x) = x \otimes 1 + 1 \otimes x + x'$ where $x' \in B(q) \otimes B(q)$. The proof of (iv) follows from these two facts via standard arguments (see Lemma 3:1:2 of **[19]**).

Now pick $a \in P_{2n}(H_*(X; \mathbf{Z}/p))$ such that $\langle a, \bar{y} \rangle \neq 0$. The argument used to prove (ii), if dualized, shows that $\langle a, B(q) \rangle = 0$ for any choice of a. We now conclude from Lemma 2:11 plus (i)–(iv) that $Q_l(a) \neq 0$. Since $\bar{y}$ and a are arbitrary we conclude, by dualizing, that $Q^{2n} = Q_l Q^{2n - 2p^l + 1}$. This concludes our proof of Theorem 2:6. □

We now prove a corollary of 2:6. Let ζ: $H^*(X; \mathbf{Z}/p) \to H^*(X; \mathbf{Z}/p)$ be the Frobenius map defined by $\zeta(x) = x^p$ for all $x \in H^*(X; \mathbf{Z}/p)$. Then $\zeta(H^*(X; \mathbf{Z}/p))$ is a sub-Hopf algebra of $H^*(X; \mathbf{Z}/p)$. By 2:5, $\zeta(H^*(X; \mathbf{Z}/p)) = T(V)$ for some set V. Fix $n \geqslant 0$. We will prove

COROLLARY 2:12. *We can select V such that the elements of V in dimension $\geqslant 2p^n$ lie in the image of Q_n.*

(When we say $x^p = Q_n(y)$ we are thinking of $\zeta(H^*(X; \mathbf{Z}/p))$ as a sub-Hopf algebra of $H^*(X; \mathbf{Z}/p)$. Thus y can be any element in $H^*(X; \mathbf{Z}/p)$.) We first describe how to obtain algebra generators for $\zeta(H^*(X; \mathbf{Z}/p))$.

LEMMA 2:13. *If a set $W \subset H^*(X; \mathbf{Z}/p)$ projects to a basis of Q^{even} then $\zeta(W)$ generates $\zeta(H^*(X; \mathbf{Z}/p))$ as an algebra.*

PROOF. Suppose we have a set W which projects to a basis of Q^{even}. Let Z be the image of W in the quotient algebra $A = H^*(X; \mathbf{Z}/p)/\zeta(H^*(X; \mathbf{Z}/p))$. Then, by definition, $A = E(Y) \otimes T(Z)$ as an algebra for some set Y. By 2:5 the Frobenius map factors through the quotient map $H^*(X; \mathbf{Z}/p) \to A$. That is, we have a commutative diagram

$$\begin{array}{ccc} H^*(X; \mathbf{Z}/p) & \xrightarrow{\zeta} & \zeta(H^*(X; \mathbf{Z}/p)) \\ \downarrow & \nearrow \zeta' & \\ A & & \end{array}$$

where ζ' is uniquely determined. Since ζ is surjective, ζ' is also surjective. Thus, using the map ζ', we can regard $\zeta(H^*(X; \mathbf{Z}/p))$ as a quotient algebra of $A = E(Y) \otimes T(Z)$. Since the elements of Y map trivially under ζ' (they are odd dimensional), it follows that $\zeta(H^*(X; \mathbf{Z}/p))$ is actually a quotient algebra of $T(Z)$. It now follows that $\zeta(W)$ generates $\zeta(H^*(X; \mathbf{Z}/p))$ as an algebra. □

LEMMA 2:14. *There exists a set $W \subset H^*(X; \mathbf{Z}/p)$ projecting to a basis of Q^{even} such that the elements of $\zeta(W)$ in dimension $\geqslant 2p^n$ lie in the image of Q_n.*

PROOF. Consider dimension $2s$. If $s \neq p^k + \cdots + p + 1$ for any k, then by 2:5 we can pick the elements of W from kernel ζ. If $s = p^k + \cdots + p + 1$ for $k \geqslant n$, then by 2:6 we can select the elements of W from Image Q_n. And if $y = Q_n(x)$ it follows that $y^p = Q_n(xy^{p-1})$. If $s = p^{n-1} + \cdots + p + 1$, then by 2:6 we can select the elements of W from Image Q_{n-1}. And if $y = Q_{n-1}(x)$, it follows that

$$y^p = \mathcal{P}^{p^{n-1}+\cdots+p+1}(y) = \mathcal{P}^{p^{n-1}+\cdots+p+1}Q_{n-1}(y) = Q_n\mathcal{P}^{p^{n-2}+\cdots+p+1}(y).$$

The last equality follows from 2:8. (The term $Q_{n-1}\mathcal{P}^{p^{n-1}+\cdots+p+1}$ disappears since $\mathcal{P}^{p^{n-1}+\cdots+p+1}$ acts trivially in dimensions $< 2(p^{n-1} + \cdots + p + 1)$.)

Corollary 2:12 now follows from the previous two lemmas. For, having obtained W as in 2:14, we can eliminate any extra algebra generators from $\zeta(W)$ and reduce to V. □

3. Bockstein spectral sequence. In this section we will discuss the Bockstein spectral sequence $\{B^r\}$ arising from the exact sequence (T_n) in §1. In particular we will give a precise statement of what we need to prove in order to show that $B^2 = B^\infty$. The Bockstein spectral sequence satisfies the following properties:

(3:1) $B^1 = H_*(X; \mathbf{Z}/p)$ with differential $d^1 = Q_n$.

(3:2) $B^\infty = k(n)_*(x)/T_n + v_n k(n)_*(X)$ where $T_n =$ the v_n torsion submodule of $k(n)_*(X)$.

(3:3) Image $d^s = \rho_n T_n(s)$ where $T_n(s) = \{x \in T_n | v_n^s x = 0\}$.Thus the sth differential d^s detects the v_n torsion elements of order s which generate direct summands in $k(n)_*(X)$ (as a $\mathbf{Z}/p[v_n]$ module). In particular $B^2 = B^\infty$ implies $v_n x = 0$ for all $x \in T_n$.

Let $N =$ rank B^2 (as a $\mathbf{Z}/p$ module). Then we have the inequalities

$$\begin{aligned} N &\geqslant \text{rank } B^\infty \quad (\text{as a } \mathbf{Z}/p \text{ module}) \\ &= \text{rank } k(n)_*(X)/T_n \quad (\text{as a } \mathbf{Z}/p[v_n] \text{ module}) \\ &= \text{rank } K(n)_*(X) \quad \left(\text{as a } \mathbf{Z}/p[v_n, v_n^{-1}] \text{ module}\right). \end{aligned}$$

Furthermore, we have equalities if and only if rank $K(n)_*(X) \geqslant N$. In other words

(3:4) $B^2 = B^\infty$ if and only if rank $K(n)_*(X) \geqslant N$.

The next three sections will be spent in proving rank $K(n)_*(X) \geq N$. Obviously, we must first determine N. Choose a Borel decomposition $\otimes A_i$ of $H^*(X; \mathbf{Z}/p)$. Let

a = the number of odd-dimensional generators,

b = the number of even-dimensional generators having dimension $< 2p^n$ plus the number of nonzero pth powers of generators having dimensions $< 2p^n$.

Regarding b, observe that it is possible to count a generator but not its pth power. We will spend the rest of this section in proving

LEMMA 3:5. $N = 2^a p^b$.

By the duality between $H_*(X; \mathbf{Z}/p)$ and $H^*(X; \mathbf{Z}/p)$ as differential Q_n Hopf algebras, it suffices to calculate the homology of $H^*(X; \mathbf{Z}/p)$ with respect to Q_n. We will use the biprimitive spectral sequence arguments of **[4]**. Given a connected Hopf algebra over $\mathbf{Z}/p$, let $\bar{A}$ be the elements of positive dimension and let $\psi^k\colon A \to \otimes_{i=1}^k A$ be the map defined by the recursive formula $\psi^2 = \bar{\psi}$, the reduced comultiplication, while $\psi^k = (\psi^{k-1} \otimes 1)\bar{\psi}$. We can define a decreasing filtration $\{F_n(A)\}$ of A by the rule

$$F_0(A) = A, \qquad F_n(A) = \bar{A}^n, \quad \text{the } n\text{-fold decomposables of } A, \text{ for } n > 0.$$

Let $E_0(A)$ be the associated graded Hopf algebra. We can define an increasing filtration $\{F^n(A)\}$ by the rule

$$F^0(A) = \mathbf{Z}/p, \qquad F^n(A) = \text{kernel } \psi^{n+1} \quad \text{for } n > 0.$$

Let $E^0(A)$ be the associated graded Hopf algebra. We define the biprimitive form of A to be E^0E_0A. The Hopf algebra E^0E_0A is primitively generated and is isomorphic, as an algebra, to $E(X) \otimes T(Y)$ for some sets X, Y. For any Borel decomposition $\otimes A_i$ of A with generators $\{a_i\}$, the set $X \cup Y$ is represented by the elements $\{a_i\}$ plus the nontrivial iterated pth powers (see 2:7 of **[4]**).

Thus, to prove 3:5, we want to show

(3:6) the biprimitive form of $H(A)$ for the case $A = H^*(X; \mathbf{Z}/p)$, $d = Q_n$ is of the form $E(X) \otimes T(Y)$ where

(i) $X^\# = a$,

(ii) $Y^\# = b$.

Associated with the differential Hopf algebra (A, d) is a spectral sequence $\{E_r\}$ where

(3:7) (i) E_1 = the biprimitive form of A,

(ii) E_∞ = the biprimitive form of $H(A)$.

We use the spectral sequence of 3:7 to prove 3:6. Consider the spectral sequence for the case $(H^*(X; \mathbf{Z}/p), Q_n)$. Each term in the spectral sequence is obviously of the form $E_r = E(X_r) \otimes T(Y_r)$ for sets X_r, Y_r. For $E_1 = E^0E_0(H^*(X; \mathbf{Z}/p))$, the set X_1 can be represented by odd-dimensional indecomposables of $H^*(X; \mathbf{Z}/p)$ while Y_1 can be represented by even-dimensional indecomposables of $H^*(X; \mathbf{Z}/p)$ plus their nonzero pth powers. This follows from 2:5. By the type of argument in 4:4 of **[4]**, we can select the elements of Y_1 to be permanent cycles in the spectral sequence. Furthermore, those in dimension $\geq 2p^n$ will eventually become boundaries in the spectral sequence. This follows from 2:6 and 2:12. It now follows that each E_r is isomorphic as a differential Hopf algebra to a tensor product $\otimes E_i$ where each E_i is one of the following types: $K = E(x)$, $L = T(y)$, $M = E(x) \otimes T(y)$, $d(x) = y$ ($|x|$ odd, $|y|$ even). Since $H(M) = E(xy^{p-1})$ it follows that $X_r^\# = X_{r+1}^\#$ for all r. Thus $X_\infty^\# = X_1^\# = a$ and 3:6(i) is established. Regarding 3:6(ii), every element of Y_1 in dimension $\geqslant 2p^n$ eventually appears in a factor of type M while the elements of Y_1 in dimension $< 2p^n$ always appear in factors of type L. It follows that $Y_\infty^\# = b$. □

4. Structure of $H_*(\Omega X; \mathbf{Z}/p)$. Let ΩX be the loop space of X. In this section, we summarize some results from **[13]**. We will use our structure theorem describing $H^*(X; \mathbf{Z}/p)$ as a Hopf algebra over $A^*(p)$ to deduce results about the Hopf algebra structure of $H_*(\Omega X; \mathbf{Z}/p)$. The Hopf algebra structure on $H_*(\Omega X; \mathbf{Z}/p)$ is the bicommutative, biassociative one induced by Ωu and the diagonal map $\Delta: \Omega X \to \Omega X \times \Omega X$. The relationship between the Hopf algebra structures on $H^*(X; \mathbf{Z}/p)$ and $H_*(\Omega X; \mathbf{Z}/p)$ is established via an Eilenberg-Moore spectral sequence converging to $H^*(\Omega X; \mathbf{Z}/p)$. It is a second quadrant spectral sequence $\{E_r, d_r\}$ of bicommutative, biassociative bigraded Hopf algebras where:

(4:1) $E_2 = \mathrm{Tor}_{H^*(X;\, \mathbf{Z}/p)}(\mathbf{Z}/p; \mathbf{Z}/p)$ as Hopf algebras.

(4:2) $E_\infty = E^0(H^*(X; \mathbf{Z}/p))$ is isomorphic to $H^*(\Omega X; \mathbf{Z}/p)$ as coalgebras where $E^0(H^*(X; \mathbf{Z}/p))$ is the bigraded Hopf algebra associated to a filtration on $H^*(\Omega X; \mathbf{Z}/p)$.

(4:3) d_r is of bidegree $(r, -r+1)$.

Regarding 4:1 the E_2 term can be calculated from any Borel decomposition $\otimes A_i$ of $H^*(X; \mathbf{Z}/p)$. Given a Borel decomposition let Y be the set of odd-dimensional generators and Z be the set of even-dimensional generators. Then

(4:4) $E_2 = E(sZ) \otimes \Gamma(sY) \otimes \Gamma(tZ)$ where sx has bidegree $(-1, |x|)$ and tx has bidegree $(-2, p^n|x|)$ if x is of height p^n. In particular the elements $sY \cup sZ$ establish an isomorphism

$$s\colon Q(H^*(X; \mathbf{Z}/p)) \simeq \mathrm{Tor}^{-1,*}_{H^*(X;\, \mathbf{Z}/p)}(\mathbf{Z}/p; \mathbf{Z}/p). \tag{4:5}$$

Since the elements of the -1 stem are permanent cycles in the spectral sequence it follows from 4:5 that the loop map $\Omega^* : Q(H^*(X; \mathbf{Z}/p)) \to P(H^*(\Omega X; \mathbf{Z}/p))$ has an obvious definition in terms of the spectral sequence.

Regarding differentials, the only differential which acts nontrivially is d_{p-1}. Using 4:5 the differential d_{p-1} can be characterized in terms of the Steenrod powers $Q_0\mathcal{P}^n$ acting on $Q(H^*(X; \mathbf{Z}/p))$, namely $d_{p-1}\gamma_p sx = sQ_0\mathcal{P}^n x$ if $|x| = 2n+1$. Thus the action of d_{p-1} is entirely determined by 2:1. In particular E_{p-1} can be written as a tensor product $\otimes E_i$ of differential Hopf algebras where each E_i is one of the following types: $K = \Gamma(x) \otimes E(y)$, $d\gamma_p(x) = y$, $L = \Gamma(x)$. Calculating the coalgebra structure of $H(E_{p-1}) = E_p = E_\infty = H^*(\Omega X\colon \mathbf{Z}/p)$ and dualizing to the algebra structure of $H_*(\Omega X; \mathbf{Z}/p)$ we conclude that:

PROPOSITION 4:6. *$H_*(\Omega X; \mathbf{Z}/p) = \mathbf{Z}/p[\chi_1]/I \otimes \mathbf{Z}/p[\chi_2]$ where I is the ideal generated by $\{x_i^p | x_i \in \chi_1\}$.*

Moreover we have an explicit relationship between the elements of χ_1 and those in $Q(H^*(X; \mathbf{Z}/p))$.

PROPOSITION 4:7. *Letting* $\chi_1 = \{x_i\}$ *there exist elements* $\{a_i\}$ *in* $Q^{\text{odd}}(H^*(X; \mathbf{Z}/p))$ *such that*

(i) $\{\beta_p \mathcal{P}^{n_i}(a_i)\}$ *is a basis of* $Q^{\text{even}}(H^*(X; \mathbf{Z}/p))$ $(|a_i| = 2n_i + 1)$,

(ii) $\langle x_i, \Omega^*(a_j)\rangle = \delta_{ij}$ (*the Kronecker delta*).

5. The structure of $BP_*(\Omega X)$, $k(n)_*(\Omega X)$, **and** $K(n)_*(\Omega X)$. In this section we will study the algebra structure of $BP_*(\Omega X)$, $k(n)_*(\Omega X)$, and $K(n)_*(\Omega X)$. We will concentrate on $BP_*(\Omega X)$. The results for $k(n)_*(\Omega X)$ and $K(n)_*(\Omega X)$ will be simple consequences. Each of the theories are multiplicative. The multiplications are related by a commutative diagram

$$\begin{array}{ccccc} BP \wedge BP & \rightarrow & k(n) \wedge k(n) & \rightarrow & K(n) \wedge K(n) \\ \downarrow & & \downarrow & & \downarrow \\ BP & \rightarrow & k(n) & \rightarrow & K(n) \end{array}$$

(By **[26]** the outside square and the right-hand square commute. It follows that the left-hand square commutes as well. For if the left hand did not commute then the obstruction $\in k(n)^*(BP \wedge BP)$ would map nontrivially to $K(n)^*(BP \wedge BP)$.) The multiplications for BP and $K(n)$ are commutative, associative, and induce Kunneth formulas (see **[17]** and **[26]**). It is not known whether $k(n)$ satisfies similar properties in general. However, since $BP_*(\Omega X) \rightarrow k(n)_*(\Omega X)$ is surjective, $k(n)$ must satisfy these properties for the space ΩX. Thus, the map $\Omega\mu$: $\Omega(X \times X) \rightarrow \Omega X$ induces a commutative, associative algebra structure on each of $BP_*(\Omega X)$, $k(n)_*(\Omega X)$ and $K(n)_*(\Omega X)$.

Our arguments for $BP_*(\Omega X)$ are extensions of those in **[15]** and **[16]**. We have surjective maps $BP_*(\Omega X) \xrightarrow{T} H_*(\Omega X) \xrightarrow{\rho} H_*(\Omega X; \mathbf{Z}/p)$. (If we think of $H_*(\Omega X)$ and $H_*(\Omega X; \mathbf{Z}/p)$ as the BP theories $BP(0)_*(\Omega X)$ and $BP(-1)_*(\Omega X)$ then T and ρ can be identified with the maps $\rho(0, \infty)$ and $\rho(-1, 0)$ of §7.) The maps T and ρ are surjective and kernel $T = (v_1, v_2, \ldots)$ while kernel $T = (p, v_1, v_2, \ldots)$. Thus we obtain $H_*(\Omega X)$ and $H_*(\Omega X; \mathbf{Z}/p)$ from $BP_*(\Omega X)$ by factoring out the ideals $(v_1, v_2, \ldots)$ and $(p, v_1, v_2, \ldots)$, respectively. Let $\hat{\chi} = \hat{\chi}_1 \cup \hat{\chi}_2$ be a set of representatives in $BP_*(\Omega X)$ for the elements $\chi = \chi_1 \cup \chi_2$ in $H_*(\Omega X; \mathbf{Z}/p)$. Let D be the set of monomials in the elements of $\hat{\chi}$ of weight $\geqslant 2$ which do not include the pth power of any element from $\hat{\chi}_1$. Then $\hat{\chi} \cup D$ is a $\Lambda = BP_*(\text{pt})$ basis of $BP_*(\Omega X)$. In fact it follows from 4:6 that

PROPOSITION 5:1. $BP_*(\Omega X)$ *is isomorphic, as an algebra, to* $\Lambda[\hat{\chi}]/J$ *where* J *is the ideal generated by* $\{R_X | x \in \hat{\chi}_1\}$ *and each* R_X *is of the form* $R_X = X^p - \sum \lambda_i X_i - \sum \omega_j d_j$ *for some* $X_i \in \hat{\chi}$, $d_j \in D$, *and* $\lambda_i, \omega_j \in \Lambda$.

(Therefore, J defines the relation by which monomials in $\hat{\chi}$ involving pth powers of elements from $\hat{\chi}_1$ can be written in terms of $\hat{\chi} \cup D$.) We have much more precise information on the relation $\{R_X\}$. The rest of our study of $BP_*(\Omega X)$ will consist in obtaining this information. First of all, it follows from 4:6 that, for each $X \in \hat{\chi}_1$,

$$X^p = pU \bmod (v_1, v_2, \ldots) \tag{5:2}$$

for some $U \in BP_*(\Omega X)$. Moreover, the elements $\{U\}$ satisfy an additional property. For each X_i and corresponding U_i, let $x_i = \rho T(X_i)$ and $u_i = \rho T(U_i)$. Let $\{a_i\}$ be the elements in $Q^{\text{odd}}(H^*(X; \mathbf{Z}/p))$ from 4:7. If $|a_i| = 2n_i + 1$, let $b_i = \mathcal{P}^{n_i}(a_i)$.

PROPOSITION 5:3. $\langle u_i, \Omega^*(b_j)\rangle = \delta_{ij}$ *(the Kronecker delta)*.

PROOF. Let $c_j = \Omega^*(a_j)$ and let $(\Omega\Delta)^p : H_*(\Omega X; \mathbf{Z}/p) \to \otimes_{i=1}^p H_*(\Omega X; \mathbf{Z}/p)$ be the p-fold reduced comultiplication defined as in §4. Then

$$\langle u_i, \Omega^*(b_j)\rangle = \langle u_i, \Omega^*\mathcal{P}^{n_j}(a_j)\rangle = \langle u_i, \mathcal{P}^{n_j}(c_j)\rangle$$
$$= \langle u_i, c_j^p\rangle = \langle \Omega\Delta^p(u_i), c_j \otimes \cdots \otimes c_j\rangle.$$

To show that the last pairing is equal to δ_{ij} let B_s be the sub-Hopf algebra of $H_*(\Omega X; \mathbf{Z}/p)$ generated by $\sum_{i \leqslant s} H_i(\Omega X; \mathbf{Z}/p)$. It follows from the implication arguments which appear in [5] that

(*) if $|x_i| = 2n_i$ then $(\Omega\Delta)^p(u_i) = x_i \otimes \cdots \otimes x_i$ in $\otimes_{i=1}^p H_*(\Omega X; \mathbf{Z}/p)//B_{2n_i-1}$.

Now, to prove 5:3, we may obviously assume that $|u_i| = |\Omega^*(b_j)|$ $(= 2pn_i)$. It follows that $|x_i| = |c_j|$ $(= 2n_i)$. Thus, by 3:10 of [23]

$$\langle B_{2n_i-1}, c_j\rangle = 0, \tag{**}$$

for c_j is primitive while any element of B_{2n_i-1} in dimension $2n_i$ is decomposable. Finally, by 4:7(ii),

$$\langle x_i, c_j\rangle = \delta_{ij}. \tag{***}$$

It follows from (*), (**), and (***) that we have the equality

$$\langle(\Omega\Delta)^p(u_i), c_j \otimes \cdots \otimes c_j\rangle = \delta_{ij}.$$

This concludes the proof of 5:3. □

We now further expand the identity 5:2. The argument uses cohomology operations. Let $\Delta_s = (0, \ldots, 0, 1, 0, \ldots)$ be the sequence with 1 in the sth position. Let $\{r_{\Delta_s}\}_{s \geqslant 1}$ be the operations in $BP^*(BP)$ defined by Quillen (see [2] or [25]). For any space X these operations act on the left of $BP^*(X)$ as differentials. They act on $\Lambda = BP_*(\text{pt})$ by the rule that

$$r_{\Delta_i} v_j = p\delta_{ij} \bmod (v_1, v_2, \ldots). \tag{5:4}$$

This follows from a knowledge of how r_{Δ_i} acts on $\Lambda \otimes_{\mathbf{Z}} Q = Q[m_1, m_2, \ldots]$ (see [29]) plus the Hazewinkel definition of v_j in terms of the elements $\{m_s\}$ (see [9]). For each sequence Δ_s we have the Milnor element $\mathcal{P}^{\Delta_s}$ in $A^*(p)$. The operations r_{Δ_s} and $\mathcal{P}^{\Delta_s}$ are related via the following diagram:

$$\begin{array}{ccc} BP_*(\Omega X) & \xrightarrow{r_{\Delta_s}} & BP_*(\Omega X) \\ \downarrow \rho T & & \downarrow \rho T \\ H_*(\Omega X; \mathbf{Z}/p) & \xrightarrow{-\mathcal{P}^{\Delta_s}} & H_*(\Omega X; \mathbf{Z}/p) \end{array} \tag{5:5}$$

(see [15] for a proof).

If we work modulo $(v_1, v_2, \ldots)^2$, then we can extend 5:2. For each $X \in \hat{\chi}_1$,

$$X^p \equiv pU - \sum_{s>1} v_s r_{\Delta_s}(U) + \sum v_s\, d_s \bmod (v_1, v_2, \ldots)^2 \tag{5:6}$$

where d_s is decomposable.

(We note that the right-hand of 5:6 is a finite sum since $BP_i(X) = 0$ for $i < 0$.) The proof of 5:6 is by induction on s. The case $s = 1$ is Proposition 5:2 of **[15]**. The general case is proved by an argument analogous to that used in proving 5:2 of **[15]**. We use 5:4 and replace the use of the operation r_{Δ_1} and the coefficient v_1 by the operation r_{Δ_s} and the coefficient v_s.

The rest of our study of $BP_*(\Omega X)$ will consist in showing that if we take the expansion 5:6, pass to $Q(BP_*(\Omega X))$ and project down to $Q(H_*(\Omega X; \mathbf{Z}/p))$, then:

PROPOSITION 5:7. *For each $n \geq 1$ the set $\{r_{\Delta_n}(U) \mid |U| \geq 2p^n\}$ projects to a linearly independent set in $Q(H_*(\Omega X; \mathbf{Z}/p))$.*

Our proof of 5:7 is analogous to the arguments used in **[16]**. Let $Q = Q(H^*(X; \mathbf{Z}/p))$ and $K = Q \cap \text{kernel}\, \beta_p$. For each U_i, let $u_i = \rho T(U_i)$. We will show

$$\langle u_i, \Omega^* K \rangle = 0 \quad \text{for each } u_i. \tag{5:8}$$

Thus, there is a well-defined pairing between the elements $\{u_i\}$ and the elements in $Q' = Q/K$. By 2:1 Q' has nontrivial elements only in odd dimension. A $\mathbf{Z}/p$ basis of Q' is represented by any set $\{b_j\} \subset Q^{\text{odd}}$ such that $\{\beta_p(b_j)\}$ is a $\mathbf{Z}/p$ basis of Q^{even}. Thus 5:3 produces a $\mathbf{Z}/p$ basis $\{b_j\}$ of Q' such that $\langle u_i, b_j \rangle = \delta_{ij}$. We will show

(5:9) for each $n \geq 1$ the map $Q \xrightarrow{\mathcal{P}^{\Delta_n}} Q \to Q'$ is surjective in dimensions $\geq 2p^n + 1$.

Thus, in dimension $\geq 2p^n + 1$, the basis $\{b_j\}$ can be chosen to be of the form $\{\mathcal{P}^{\Delta_n}(h_j)\}$. Therefore, when we pass to ΩX, it follows that in dimensions $\geq 2p^n$ we have the identities

$$\langle \mathcal{P}^n(u_i), \Omega^* h_j \rangle = \langle u_i, \mathcal{P}^{\Delta_n}\Omega^*(h_j) \rangle = \langle u_i, \Omega^* \mathcal{P}^{\Delta_n}(h_j) \rangle = \langle u_i, \Omega^* b_j \rangle = \delta_{ij}$$

(the last equality follows from 5:3). Thus the set $\{\mathcal{P}^{\Delta_n}(u_i)\}$ is linearly independent in dimension $\geq 2p^n$. By 5:5 this suffices to prove 5:7. So we are left with proving 5:8 and 5:9.

PROOF OF 5:8. First of all we show

($*$) the map ρ: $Q(H^*(X)) \to Q$ maps onto K.

That is, each element of K has a representative in $H^*(X; \mathbf{Z}/p)$ which survives the p torsion Bockstein cohomology spectral sequence $\{B_r\}$. Since $B_2 = B_\infty$ (see 4:6:1 of **[19]**) and $d_1 = \beta_p$ it suffices to show that each element of K has a representative in $H^*(X; \mathbf{Z}/p)$ on which β_p acts trivially. Consider the biprimitive spectral sequence $\{E_r\}$ of 3:7 for $(A, d) = (H^*(X; \mathbf{Z}/p), \beta_p)$. By the discussion in §3 the elements of Q define unique elements in $E_1 = E^0E_0H^*(X; \mathbf{Z}/p)$. It suffices to show that the elements of K survive the spectral sequence. It follows from 2:6 that E_1 is isomorphic, as a differential Hopf algebra, to a tensor product $\bigotimes E_i$ of differential Hopf algebras where each E_i is one of the following type: $K = E(x) \otimes T(y)$, $dx = y$, $L = E(x)$. Thus E_2 is an exterior algebra and the elements of K define nonzero elements in E_2. By the usual Hopf algebra arguments (see **[4]**) E_2 exterior implies $E_2 = E_\infty$. In particular, K survives to E_∞.

We use ($*$) to prove 5:8. Pick $k \in K$. Then $k = \rho(l)$ for some $l \in Q(H^*(X))$ and

$$\langle u_i, \Omega^*(k) \rangle = \langle u_i, \rho\Omega^*(l) \rangle = \langle \rho T(u_i), \rho\Omega^*(l) \rangle = \rho\langle T(u_i), \Omega^*(l) \rangle.$$

Thus it suffices to show that $\langle T(u_i), \Omega^*(l)\rangle = 0$. Now $\langle T(u_i), \Omega^*(l)\rangle \in \mathbf{Z}_{(p)}$. Hence it is equivalent to show that $p\langle T(u_i), \Omega^*(l)\rangle = 0$. And

$$p\langle T(u_i), \Omega^*(l)\rangle = \langle pT(u_i), \Omega^*(l)\rangle = \langle T(X_i)^p, \Omega^*(l)\rangle = 0.$$

The second equality follows from 5:2. The third equality follows from 3:10 of **[23]**. For $T(X_i)^p$ is indecomposable while $\Omega^*(l)$ is primitive. □

PROOF OF 5:9. By 2:6, $Q_n = Q_0\mathcal{P}^{\Delta_n}Q_0$ maps onto Q^{even} in dimensions $\geqslant 2p^n + 2$. By 2:2, $\mathcal{P}^{\Delta_n}$ is trivial when restricted to Q^{even}. Thus $Q_0\mathcal{P}^{\Delta_n}$ maps onto Q^{even} in dimensions $\geqslant 2p^n + 2$. Since $Q_0 = \beta_p$, 5:9 follows. □

We conclude this section by using our results for $BP_*(\Omega X)$ to deduce results about the algebra structures of $k(n)_*(\Omega X)$ and $K(n)_*(\Omega X)$. We can obtain $k(n)_*(\Omega X)$ from $BP_*(\Omega X)$ by factoring out the ideal $I_n = (p, v_1, \ldots, v_{n-1}, v_{n+1}, \ldots)$. And we can obtain $K(n)_*(\Omega X)$ from $k(n)_*(\Omega X)$ by localizing with respect to v_n. Thus any element of $BP_*(\Omega X)$ defines unique elements in $k(n)_*(\Omega X)$ and in $K(n)_*(\Omega X)$. We will use the same symbol to denote these elements. From 5:1 it follows that:

PROPOSITION 5:10. *For $h = k(n)$ or $K(n)$, $h_*(\Omega X)$ is isomorphic, as an algebra, to $h_*(\text{pt})[\hat{\chi}]/J$ where $\hat{\chi}$ and J are as in* 5:1.

The relations $\{R_X\}$ generating J must be of the following form:

$$\begin{aligned}&\text{if } |X^p| < 2p^n \text{ then } R_X = X^p,\\ &\text{if } |X^p| \geqslant 2p^n \text{ then } R_X = X^p - p\overline{X} \text{ for some } \overline{X} \in h_*(\Omega X).\end{aligned} \tag{5:11}$$

Since we have a commutative diagram

$$\begin{array}{ccccc} BP_*(\Omega X) & \to & k(n)_*(\Omega X) & \to & H_*(\Omega X; \mathbf{Z}/p) \\ \downarrow & & \downarrow & & \downarrow \\ Q(BP_*(\Omega X)) & \to & Q(k(n)_*(\Omega X)) & \to & Q(H_*(\Omega X; \mathbf{Z}/p)) \end{array}$$

it follows from 5:6 and 5:7 that

PROPOSITION 5:12. *For $h = k(n)$ the elements $\{\overline{X}\}$ from* 5:11 *project to a linearly independent set in $Q(H_*(\Omega X; \mathbf{Z}/p))$.*

6. Eilenberg-Moore spectral sequences. In this section we will use the results of §§4 and 5 to complete the proof of Theorem 1:1. By 3:4 and 3:5 we need only show

$$\operatorname{rank} K(n)_*(X) \geqslant N \quad \text{where } N = 2^a p^b. \tag{6:1}$$

Our main tool in proving 6:1 will be Eilenberg-Moore type spectral sequences. For each of the theories $h = H\mathbf{Z}/p$, $k(n)$, and $K(n)$, there is a 1st and 4th quadrant spectral sequence $\{E^r(h)\}$ of $h_*(\text{pt})$ modules satisfying

$$E^2(h) = \operatorname{Tor}^{h_*(\Omega X)}(h_*(\text{pt}); h_*(\text{pt})), \tag{6:2}$$

$$E^\infty(h) = E^0(h_*(X)), \tag{6:3}$$

$$d^r \text{ is of bidegree } (-r, r-1). \tag{6:4}$$

In each case the spectral sequence arises from the fact that X has the same homotopy type as $B_{\Omega X}$, the classifying space of ΩX. The space $B_{\Omega X}$ is filtered by an increasing sequence $\{B_n\}$ where B_n is the n-fold projective space of ΩX. This induces a fltration of $h_*(B_{\Omega X})$ and, hence, a spectral sequence $\{E^r(h)\}$. Properties 6:3 and 6:4 follow from the general properties of the construction. Regarding property 6:2, it follows if h is a commutative, associative, multiplicative theory, satisfies a Kunneth formula and $h_*(\Omega X)$ is torsion free (see §4 of **[21]**). By the discussion at the beginning of §5, property 6:2 holds for $H\mathbf{Z}/p$ and $K(n)$. It also holds for $k(n)$. For we can set up the spectral sequence for *BP* and, by the above reasoning, 6:2 will hold for this spectral sequence. Then we can reduce from *BP* to $k(n)$. The natural maps $H\mathbf{Z}/p \leftarrow k(n) \rightarrow K(n)$ induce maps $\{E^r(H\mathbf{Z}/p)\} \leftarrow \{E^r(k(n))\} \rightarrow \{E^r(K(n))\}$ between the spectral sequences. The second map is simply localization with respect to v_n. These maps play a key role in our proof of 6:1. We want to study the spectral sequence $\{E^r(K(n))\}$. The spectral sequence $\{E^r(H\mathbf{Z}/p)\}$ has been studied in detail (see **[7]** or **[13]**). Using the above maps, this information is used to determine the differentials in $\{E^r(k(n))\}$ and then in $\{E^r(K(n))\}$.

We begin by considering the E^2 terms in these spectral sequences. From now on we will use Tor(h) to denote $\mathrm{Tor}^{h_*(\Omega X)}(h_*(\mathrm{pt}), h_*(\mathrm{pt}))$. It follows from 4:6 and 5:10 that:

PROPOSITION 6:5. *For* $h = H\mathbf{Z}/p$, $k(n)$, *or* $K(n)$, Tor(h) *is isomorphic to the homology of the complex* $E(s\chi) \otimes \Gamma(t\chi_1) \otimes h_*(\mathrm{pt})$ *where* sX *has bidegree* $(1, |X|)$, tX *has bidegree* $(2, 2p|X|)$, *and the differential* d *acts by the rule* $dsX = 0$, $d\gamma_s(tX) = \gamma_{s-1}(tX)Q_X$. *Here* Q_X *is determined from* R_X *of* 5:1 *by the rule* $Q_X = \sum \lambda_i sX_i$.

(See §7 of **[15]** as well as §§3 and 8 of **[24]** for a justification of this theorem. In **[15]** we showed that $\mathrm{Tor}^{BP_*(\Omega X)}(BP_*(\mathrm{pt}); BP_*(\mathrm{pt}))$ was the homology of the complex $E(s\chi) \otimes \Gamma(t\chi_1)$ with $BP_*(\mathrm{pt})$ coefficients. So now we are simply changing our coefficient rings.) The results of §5 give strong restrictions on the elements $\{R_X\}$. We now use this information to impose further restrictions on Tor(h). The elements $s\chi$ define an isomorphism

$$s\colon Q\big(h_*(\Omega X)\big) \simeq \mathrm{Tor}_{1,*}(h). \tag{6:6}$$

Furthermore, the isomorphisms 6:6, in the cases $h = k(n)$ and $h = H\mathbf{Z}/p$, are related by the following commutative diagram:

$$\begin{array}{ccc} Q\big(k(n)_*(\Omega X)\big) & \simeq & \mathrm{Tor}_{1,*}(k(n)) \\ \downarrow & & \downarrow \\ Q\big(H_*(\Omega X; \mathbf{Z}/p)\big) & \simeq & \mathrm{Tor}_{1,*}(H\mathbf{Z}/p) \end{array}$$

Consider the map $k(n)_*(\Omega X) \rightarrow Q(k(n)_*(\Omega X)) \simeq \mathrm{Tor}_{1,*}(k(n))$. (The first map is the quotient map while the second comes from 6:6.) Under this map R_X is mapped to Q_X. Thus the restrictions imposed on $\{R_X\}$ by 5:11 and 5:12 imply that we can rewrite the elements of $s\chi$ so that

LEMMA 6:7. *$\mathrm{Tor}(k(n))$ = the homology of the complex $A \otimes B \otimes C \otimes \mathbf{Z}/p[v_n]$ where*

$$A = \otimes E(x_i), \qquad dx_i = 0,$$
$$B = \otimes \Gamma(y_j), \qquad dy_j = 0, \qquad |y_j| < 2p^n - 2,$$
$$C = \otimes E(x_k) \otimes \Gamma(y_k), \quad d\gamma_s(y_k) = v_n \gamma_{s-1}(y_k) x_k, \quad |y_k| \geqslant 2p^n - 2.$$

REMARK. The generators $S = \{x_i\} \cup \{x_k\} \cup \{y_j\} \cup \{y_k\}$ are obtained from the set $s\chi \cup t\chi_1$ by rewriting elements.

REMARK. We are using $|x|$ to denote total dimension in the bigraded module $\mathrm{Tor}(k(n))$. We will follow the same convention for the rest of this section.

We also have the identities

LEMMA 6:8. $\mathrm{Tor}(K(n)) = A \otimes B \otimes \mathbf{Z}/p[v_n, v_n^{-1}]$.

LEMMA 6:9. $\mathrm{Tor}(H\mathbf{Z}/p) = A \otimes B \otimes C$.

Both of these results are corollaries of 6:7. As our last result on the E^2 terms of the spectral sequences we observe that, by the usual homological algebra arguments, the short exact sequence $0 \to k(n)_*(\Omega X) \xrightarrow{v_n} k(n)_*(\Omega X) \to H_*(\Omega X; \mathbf{Z}/p) \to 0$ induces the exact couple:

$$\begin{array}{ccc} \mathrm{Tor}(k(n)) & \overset{v_n}{\to} & \mathrm{Tor}(k(n)) \\ & \Delta_n \nwarrow \quad \swarrow \rho_n & \\ & \mathrm{Tor}(H\mathbf{Z}/p) & \end{array} \tag{6:10}$$

We now consider how differentials act in the Eilenberg-Moore spectral sequences. We begin with $\{E^n(H\mathbf{Z}/p)\}$. It is analogous to the spectral sequence considered in §4 and satisfies analogous properties. It is a spectral sequence of bicommutative, biassociative Hopf algebras. Moreover $E^2 = \mathrm{Tor}(H\mathbf{Z}/p)$ as Hopf algebras and $E^\infty = H_*(X; \mathbf{Z}/p)$ as coalgebras. (See [**7**]. For the last property see the proof of 2:8 of [**13**].)

Starting out from the E^2 term given by 6:9 and using the arguments of [**7**], it follows that d^r acts trivially unless $r = 2p^s - 1$ for some $s \geqslant 1$. Furthermore, $E^{2p^s-1}(H\mathbf{Z}/p)$ is isomorphic, as a differential Hopf algebra, to a tensor product of differential Hopf algebras where each factor is either a differential Hopf algebra on which $d = 0$ or a differential Hopf algebra of the form

$$M_s = E(x) \otimes \Gamma(y), \qquad d\gamma_{p^s}(y) = x.$$

Since $E^\infty(H\mathbf{Z}/p)$ is a finite $\mathbf{Z}/p$ module, it follows that

$$E^\infty(H\mathbf{Z}/p) = E(T) \otimes \left[\bigotimes_{i=1}^{l} H(M_{s_i}) \right] \text{ for some set } T \text{ and some integers } s_1, \ldots, s_l \tag{6:11}$$

(that is any divided polynomial algebra $\Gamma(y) \subset E^2(H\mathbf{Z}/p)$ will eventually appear in a factor of type M_s). The homology of M_s, $H(M_s)$, is a divided polynomial algebra truncated at height p^s. Therefore, by dualizing 6:11, we have a Borel decomposition of $H^*(X; \mathbf{Z}/p)$. By the definition of $N = 2^a p^b$ it follows that

$a = T^\sharp$

$b =$ the number of nontrivial divided pth powers which appear in the various $H(M_{s_i})$ and have total dimension $< 2p^n$.

Next, consider the spectral sequence $E^n(K(n))$. Again this is a spectral sequence of bicommutative, biassociative Hopf algebras (see **[26]** for the necessary multiplicative structure on $K(n)$ to ensure this Hopf algebra structure). Starting out from $E^2(K(n))$ as given by 6:8, and using the arguments of [7], it follows that $d^n = 0$ unless $r = 2p^s - 1$ for some $s \geq 1$. Also, $E^{2p^s-1}(K(n))$ can be written as a tensor product of differential Hopf algebras on which $d = 0$ or a differential Hopf algebra of the form $M_s \otimes \mathbf{Z}/p[v_n, v_n^{-1}]$. Since $E^\infty(K(n))$ is a finitely generated $\mathbf{Z}/p[v_n, v_n^{-1}]$ module it follows that

$$E^\infty(K(n)) = E(T') \otimes \left[\bigotimes_{j=1}^{m} H(M_{s_j})\right] \otimes \mathbf{Z}/p[v_n, v_n^{-1}] \text{ for some set } T' \tag{6:12}$$

and some integers $s_1, \ldots, s_m$.

Thus rank $E^\infty(K(n))$ (as a $\mathbf{Z}/p[v_n, v_n^{-1}]$ module) $= 2^c p^d$ where

$c = T'^\sharp$,

$d =$ the number of nontrivial divided pth powers which appear in the various $H(M_{s_j})$.

To prove 6:1 we want to show that rank $E^\infty(K(n)) \geq 2^a p^b$. Thus it suffices to show that $c \geq a$ and $d \geq b$.

LEMMA 6:13. $c = a$.

PROOF. This follows from a counting argument. We can think of the spectral sequences $\{E^r(H\mathbf{Z}/p)\}$ and $\{E^r(K(n))\}$ as having an E^1 term given by 6:7. Thus $E^1(H\mathbf{Z}/p) = A \otimes B \otimes C$ and $E^1(K(n)) = A \otimes B \otimes C \otimes \mathbf{Z}/p[v_n, v_n^{-1}]$. Hence both E^1 terms have the same number, α, of odd-dimensional generators and the same number, β, of even-dimensional generators. Next, in both spectral sequences, the number of odd-dimensional generators which disappear equals the number of even-dimensional generators. For, in $\{E^r(H\mathbf{Z}/p)\}$, odd-dimensional generators disappear only by appearing in a factor M_s while, in $\{E^r(K(n))\}$, they disappear only by appearing in a factor of C or in a factor $M_s \otimes \mathbf{Z}/p[v_n, v_n^{-1}]$. Also, as we have already argued, the number of such factors, in each case, equals the number of even-dimensional generators. We conclude that $a = \alpha - \beta = c$. □

LEMMA 6:14. $b \leq d$.

PROOF. Let

$e =$ the number of divided pth powers of $E^2(H\mathbf{Z}/p) = A \otimes B \otimes C$ which have total dimension $< 2p^n$ and survive nontrivially to $E^\infty(H\mathbf{Z}/p)$.

$f =$ the number of divided pth powers of $E^2(K(n)) = A \otimes B \otimes \mathbf{Z}/p[v_n, v_n^{-1}]$ which survive nontrivially to $E^\infty(K(n))$.

We will prove 6:14 in three steps. We wll show $b = e \leq f \leq d$.

(i) $b = e$. By the arguments of [7], we can rewrite the elements of $E^2(H\mathbf{Z}/p) = A \otimes B \otimes C$ such that the factors $E(x)$ and $\Gamma(y)$ of each M_s can be identified with factors $E(x_i)$, $\Gamma(y_j)$, $E(x_k)$, or $\Gamma(y_k)$ of $E^2(H\mathbf{Z}/p)$ which have survived to $E^{2p^s-1}(H\mathbf{Z}/p)$. Moreover, this rewriting does not disturb any of the previous structure theorems. In particular, 6:7, 6:8 and 6:9 are still valid. For rewriting the elements of A or B or C only involves other elements of A or B or C, respectively. First of all, the element x in M_s has total dimension $\equiv -1 \pmod{2p}$ while the elements $\{x_k\}$ in C have total dimension $\equiv 3 \pmod{2p}$. Therefore x comes from the elements of A. By rewriting the elements of A we can identify x with one of the $\{x_i\}$ in A. Secondly, the element y in M_s comes from an element of B if $|y| < 2p^n$. Rewriting the elements of B, we can identify y with one of the $\{y_j\}$. Thirdly, if $|y| > 2p^n$ we can rewrite the elements of C to identify y with one of the $\{y_k\}$. This canonical form we have given to the spectral sequence implies equality (i).

(ii) $e \leqslant f$. Pick a divided pth power $\gamma_{p^s}(y_t)$ of $E^2(H\mathbf{Z}/p)$ which has total dimension $< 2p^n$. By the relation between the E^2 terms given by 6:7–6:9, it follows that there are corresponding elements $\gamma_{p^s}(y_t)$ in $E^2(k(n))$ and in $E^2(K(n))$. To prove (ii) it suffices to show

(*) $d^r\gamma_{p^s}(y_t) = 0$ for all r in $\{E^2(H\mathbf{Z}/p)\}$ implies $d^r\gamma_{p^s}(y_t) = 0$ for all r in $\{E^r(K(n))\}$.

So suppose $d^r\gamma_{p^s}(y_t) = 0$ for all r in $\{E^r(H\mathbf{Z}/p)\}$. Consider $\gamma_{p^s}(y_t) \in E^2(k(n))$. By the exact triangle 6:10 it follows that $E^2(k(n)) \simeq E^2(H\mathbf{Z}/p)$ in total dimension $< 2p^n$. Since d^r lowers dimension by 1 we can prove that, for all r, $E^r(k(n)) \simeq E^r(H\mathbf{Z}/p)$ in total dimension $< 2p^n$. We work by double induction, first on r and secondly on total dimension. Thus $d^r\gamma_{p^s}(y_t) = 0$ in $E^r(H\mathbf{Z}/p)$ implies $d^r\gamma_{p^s}(y_t) = 0$ in $E^r(k(n))$. Since $E^r(K(n))$ is obtained from $E^r(k(n))$ by localizing it follows that $d^r\gamma_{p^s}(y_t) = 0$ in $E^r(K(n))$.

(iii) $f \leqslant d$. This inequality is trivial. □

7. *BP* theories. The last three sections of this paper are devoted to proving Theorem 1:5. We will work with cohomology rather than homology throughout the discussion. In §§7 and 8 we will describe some useful machinery. In §9 we will prove the theorem. In this section we describe some spectral sequences associated to various BP theories.

First of all, dual to the homology spectral sequence described in §3 is a cohomology spectral sequence $\{B_r, d_r\}$ induced by the exact couple

$$\begin{array}{ccc} k(n)^*(X) & \overset{v_n}{\rightarrow} & k(n)^*(X) \\ & \Delta_n \nwarrow \quad \swarrow \rho_n & \\ & H^*(X; \mathbf{Z}/p) & \end{array} \tag{7:1}$$

Moreover properties 3:1–3:3 are still valid when we replace homology by cohomology. By letting $k(0)^*(X) = H^*(X)$ and $v_0 = p$ we can include the case $n = 0$ in the above as well.

For $1 \leqslant n < \infty$ let $BP\langle n\rangle^*(X)$ be the BP theory where $BP\langle n\rangle^*(\mathrm{pt}) = \mathbf{Z}_{(p)}[v_1, v_2, \ldots, v_n]$. Also let $BP\langle 0\rangle^*(X)$ denote $H^*(X)$, $BP\langle -1\rangle^*(X)$ denote $H^*(X; \mathbf{Z}/p)$, and $BP\langle\infty\rangle^*(X)$ denote $BP^*(X)$. Given integers $-1 \leqslant t < s \leqslant \infty$ there is a canonical map $\rho(t, s)$: $BP\langle s\rangle^*(X) \rightarrow BP\langle t\rangle^*(X)$. For $n \geqslant 0$ there is an exact couple

$$\begin{array}{ccc} BP\langle n\rangle^*(X) & \overset{v_n}{\rightarrow} & BP\langle n\rangle^*(X) \\ \Delta_n \nwarrow & & \swarrow \rho(n-1, n) \\ & BP\langle n-1\rangle^*(X) & \end{array} \qquad (7:2)$$

where v_n is multiplication by v_n ($v_0 = p$). We can derive a Bockstein spectral sequence $\{\hat{B}_r, \hat{d}_r\}$ from this exact couple. This spectral sequence satisfies the following properties:

(7:3) $\hat{B}_1 = BP\langle n-1\rangle^*(X)$.

(7:4) $\hat{B}_\infty = BP\langle n\rangle^*(X)/T_n + v_n BP\langle n\rangle^*(X)$ where T_n is the v_n torsion submodule of $BP(n)^*(X)$.

(7:5) Image $\hat{d}_s = \rho(n-1, n)T_n(s)$ where $T_n(s) = \{x \in T_n | v_n^s x = 0\}$.

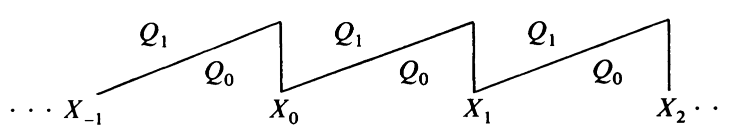
(7:6) For $n \geqslant 1$, Image $\hat{d}_s$ is represented in $\hat{B}_s$ by v_{n-1} torsion elements of $\hat{B}_1$ which have survived to $\hat{B}_s$.

See [**11**] for $BP(n)$ and all of the above properties. Also, since there is a natural map from 7:2 to 7:1 it follows that:

(7:7) There is a natural map $\{\hat{B}_r, \hat{d}_r\} \rightarrow \{B_r, d_r\}$ which agrees on the B_1 terms with the map $\rho(-1, n-1)$: $BP\langle n-1\rangle^*(X) \rightarrow H^*(X; \mathbf{Z}/p)$.

For $n = 0$ the above two spectral sequences agree and are identical with the classical Bockstein spectral sequence (see [**3**]).

8. $E(Q_0, Q_1)$ **modules.** In §9 we will also need to be able to describe how Q_0 and Q_1 act on $H^*(X; \mathbf{Z}/p)$. Since $Q_0^2 = Q_1^2 = 0$ and $Q_0Q_1 = -Q_1Q_0$ we can consider $H^*(X; \mathbf{Z}/p)$ as a graded module over the exterior algebra $E(Q_0, Q_1)$. It is very convenient to describe the action of Q_0 and Q_1 in terms of such a module structure. In particular we can use the machinery from Part III of [**2**] where Adams classified finite-dimensional $E(Q_0, Q_1)$ modules. The key to his work is the "lightning flash" module. Let x_i have dimension $2i(p-1)$ ($i \in \mathbf{Z}$). Let $L =$ the $E(Q_0, Q_1)$ module generated by $\{x_i\}$ with relations $Q_1(x_i) = Q_0(x_{i+1})$. We can display L schematically as:



Two $E(Q_0, Q_1)$ modules M and N are stably isomorphic if there exist free modules F and G such that $M \oplus F \cong N \oplus G$. Adams showed in [**2**] that the stable class of any finite-dimensional $E(Q_0, Q_1)$ module is a direct sum of stable classes of subquotient modules of L (see Theorem 16:11 of Part III of [**2**]). In the situation dealt with in §9 we will only need the following modules derived from L. For each $t \in \mathbf{Z} \cup \{\infty\}$ define the module $L(t)$ as follows.

$t < 0$: $L(t) =$ the $E(Q_0, Q_1)$ submodule of L generated by $\{x_i | t \leqslant i < 0\}$.

$t \geqslant 0$: $L(t)$ = the $E(Q_0, Q_1)$ quotient module of L formed by factoring out the submodule generated by $\{x_i | i < 0 \text{ or } i > t\}$.

$t = \infty$: $L(\infty)$ = the $E(Q_0, Q_1)$ quotient module of L formed by factoring out the submodule generated by $\{x_i | i < 0\}$.

For a schematic representation of $L(t)$ take the display of L and truncate it at both ends in the appropriate manner. The $L(t)$ modules are interrelated in the following manner. Given $q, r \in \mathbf{Z}$ then

$$L(q) \otimes L(r) \cong L(q + r) \oplus F \quad \text{where } F \text{ is a free } E(Q_0, Q_1) \text{ module.} \quad (8:1)$$

PROOF. We will only do the case $q \geqslant 0$, $r \geqslant 0$. The other cases are similar. Define a map γ: $L(q + r) \to L(q) \otimes L(r)$.

$$\gamma(x_s) = \sum_{i+j=s;\ 0 \leqslant i \leqslant q;\ 0 \leqslant j \leqslant r} x_i \otimes x_j.$$

It is easy to verify that γ is an injective map of $E(Q_0, Q_1)$ modules. If we pass to Q_0 or Q_1 homology then, in each case, γ is an isomorphism. It follows from 16:3 of Part III of [**2**] that $L(q + r)$ and $L(q) \otimes L(r)$ are stably isomorphic. More exactly $L(q + r) \oplus F \cong L(q) \otimes L(r) \oplus G$, where F = cokernel γ and G = kernel γ are free. Since γ is injective 8:1 follows. □

The following is a simple corollary of 8:1. Let m be a positive integer. Then

$$L(1)^m \cong L(m) \oplus F \quad \text{where } F \text{ is a free } E(Q_0, Q_1) \text{ module.} \quad (8:2)$$

9. Proof of Theorem 1:5. In this section we will prove Theorem 1:5. Fix an integer $m \geqslant 1$. Our study of $BP\langle 1 \rangle^*(\prod_{i=1}^m X)$ will depend on a knowledge of the structure of $H^*(\prod_{i=1}^m X; \mathbf{Z}/p)$ as an $E(Q_0, Q_1)$ module. First of all:

(9:1) $H^*(X; \mathbf{Z}/p) = E(a) \otimes E(b) \otimes T(c)$ where $|a| = 3$, $|b| = 2p + 1$, $|c| = 2p + 2$ and the action of the elements $\{Q_s\}$ on $H^*(X; \mathbf{Z}/p)$ is determined by $Q_0(b) = Q_1(a) = c$.

(For $p = 2$ we have the further relation $c = a^2$ which we ignore.) Let Σ be the $E(Q_0, Q_1)$ module with $\mathbf{Z}/p$ in dimension 1 and 0 in all other dimensions. For any integer $s \geqslant 1$ and any $E(Q_0, Q_1)$ module M define the s-fold suspension $\Sigma^s M$ to be M tensored with s copies of Σ. Then

$$H^*(X; \mathbf{Z}/p) = L(0) \oplus \Sigma^3 L(1) \oplus \Sigma^{2p^2+2p-2} L(-1) \oplus \Sigma^{2(p^2+p+1)} L(0) \oplus F$$
$$\text{where } F \text{ is a free } E(Q_0, Q_1) \text{ module.} \quad (9:2)$$

We obtain this decomposition by writing down elements of $H^*(X; \mathbf{Z}/p)$ in order of dimension.

$$(a, b, c)(ab, ac, bc, c^2) \cdots (abc^{p-2}, ac^{p-1}, bc^{p-1})(abc^{p-1}). \quad (1)$$

The brackets indicate the decomposition of $H^*(X; \mathbf{Z}/p)$ into a direct summand of $E(Q_0, Q_1)$ modules. In particular, $L(0)$ and $\Sigma^3 L(1)$ are the first two summands while $\Sigma^{2p^2+2p-2} L(-1)$ and $\Sigma^{2(p^2+p+1)}$ are the last two summands. All other summands are free.

We can extend 9:2 to a decomposition of $H^*(\prod_{i=1}^m X; \mathbf{Z}/p) = \otimes_{i=1}^m H^*(X; \mathbf{Z}/p)$ by using 8:1. In particular $H^*(\prod_{i=1}^m X; \mathbf{Z}/p)$ contains the summand $\Sigma^{3m}L(1)^m = \Sigma^{3m}L(m) \oplus F$. It is this summand which will produce the $BP\langle 1\rangle$ torsion described in 1:5. We now demonstrate this fact.

(A) *The case* $\tilde{H}^*(Y; \mathbf{Z}/p) = \Sigma^s L(m)$ *for some* $s \geq 0$. We will show that the torsion submodule of $BP\langle 1\rangle^*(Y)$ is generated by elements $\{z_1, \ldots, z_m\}$ which satisfy the relations

$$pz_m = 0, \quad v_1 z_1 = 0, \quad pz_i = v_1 z_{i+1} \qquad (1 \leq i < m). \tag{9:3}$$

Moreover, any relation satisfied by the elements $\{z_1, \ldots, z_m\}$ is a consequence of 9:3. In particular, we have the relations

$$v_1^{i-1} z_i \neq 0 \quad \text{while } v_1^i z_i = 0. \tag{9:4}$$

To see that we have generators $\{z_i\}$ satisfying 9:3 and 9:4 we will use Bockstein spectral sequence arguments. We will use the spectral sequences described in §7 in order to pass from $H^*(Y; \mathbf{Z}/p)$ to $H^*(Y)$ and then to $BP\langle 1\rangle^*(Y)$. First we pass from $H^*(Y; \mathbf{Z}/p)$ to $H^*(Y)$ using $\{\hat{B}_r\}$ for the case $n = 0$. From the fact that $\tilde{H}^*(Y; \mathbf{Z}/p) \cong L(m)$ we can easily calculate that

$$\tilde{H}^*(Y) = \mathbf{Z}_{(p)} \oplus \mathbf{Z}/p \oplus \cdots \oplus \mathbf{Z}/p \quad (m \text{ copies of } \mathbf{Z}/p). \tag{9:5}$$

Choose generators $\{y, y_1, \ldots, y_m\}$ of $\tilde{H}^*(Y)$ where y generates the free summand while $\{y_1, \ldots, y_m\}$ generate the torsion summands. Under the map $\tilde{H}^*(Y) \to \tilde{H}^*(Y; \mathbf{Z}/p)$, y maps to x_0 while y_i maps to $Q_0(x_i)$. Observe that

$$|y_i| = |y| + 2i(p-1) + 1. \tag{9:6}$$

Next we pass from $H^*(Y)$ to $BP\langle 1\rangle^*(Y)$ using $\{\hat{B}_r\}$ for the case $n = 1$. The $\hat{B}_1$ term is given by 9:5. The elements $\{y_1, \ldots, y_m\}$ lift to elements $\{z_1, \ldots, z_m\}$ in $BP\langle 1\rangle^*(Y)$. For since $\hat{d}_r$ changes degree by $2r(p-1)+1$ we can use 9:6 plus dimension arguments to show that the elements $\{y_1, \ldots, y_m\}$ are permanent cycles in the spectral sequence.

The relations 9:3. We now show that the elements $\{z_i\}$ can be chosen to satisfy 9:3. First of all, since $BP\langle 1\rangle^i(Y) \cong H^i(Y)$ in dimension $\geq |y_m|$ it follows that

$$pz_m = 0. \tag{9:7}$$

Secondly, we can choose z_1 to satisfy

$$v_1 z_1 = 0, \tag{9:8}$$

while for any choice of $\{z_i\}$ we have

$$v_1 z_i \neq 0 \quad \text{for } 2 \leq i \leq m. \tag{9:9}$$

To prove these two facts it suffices, by 7:5, to show that $y_1 \in \text{Image } \hat{d}_1$ while $y_i \notin \text{Image } \hat{d}_1$ if $2 \leq i \leq m$. Map from $\{\hat{B}_r\}$ to $\{B_r\}$ using 7:7. Then the elements $\{y_i\}$ map to $\{Q_0(x_i)\}$. However, only $Q_0(x_1)$ is hit (under Q_1) by an element which lifts to $H^*(X)$.

Thirdly, for each $1 \leq i < m$, we have the relation

$$v_1 z_{i+1} = p\bar{z} \quad \text{for some } \bar{z} \in BP\langle 1\rangle^*(Y). \tag{9:10}$$

For using the spectral sequence $\{B_r\}$ for $n = 1$ and the fact that $\tilde{H}^*(Y; \mathbf{Z}/p) \cong L(m)$, we can easily deduce that $k(1)^*(Y)$ has no higher v_1 torsion. The exact sequence.

$$BP\langle 1\rangle^*(Y) \xrightarrow{xp} BP\langle 1\rangle^*(Y) \to k(1)^*(Y)$$

then yields 9:10.

Lastly,

$$\bar{z} = z_i. \tag{9:11}$$

To prove 9:11 we need only show that $\bar{z}$ is not divisible by v_1. For it then follows from the exact sequence

$$BP\langle 1\rangle^*(Y) \xrightarrow{v_1} BP\langle 1\rangle^*(Y) \to H^*(Y)$$

that $\bar{z}$ maps nontrivially to y_i and, hence, can be chosen to be z_i. Suppose $\bar{z}$ is divisible by v_1. Let $\bar{z} = v_1 w$. By 9:10 we have $v_1(z_{i+1} - pw) = 0$. But replacing z_{i+1} by $z_{i+1} - pw$ we then contradict 9:9.

Relations 9:3 now follow from 9:8–9:10.

The relations 9:4. Relations 9:3 have the following consequences:

$$p^j z_i = v_1^j z_{i+j}. \tag{9:12}$$

(We employ the convention that $z_s = 0$ when $s < 0$ or $s > m$.) In particular, $p^i z_1 = v_1^i z_i = 0$. Thus to prove 9:4 it suffices to show that $v_1^{i-1} z_i \neq 0$. We will use the spectral sequences $\{\hat{B}_r\}$ for $n = 1$. In particular, we will use the relation between v_1 torsion and differentials given by 7:5. As already noted the $\hat{B}_1$ term is given by 9:5 and the torsion generators $\{y_1, \ldots, y_m\}$ of $\hat{B}_1$ are permanent cycles. Thus differentials can only act nontrivially on elements from the free summand of $\hat{B}_1$ generated by y. To determine exactly how differentials do act observe that the elements $\{y_1, \ldots, y_m\}$ are killed by differentials in the spectral sequence. This follows from 7:5 and the fact that $v_1^i z_i = 0$ for $1 \leqslant i \leqslant m$. It now follows from dimension arguments (see 9:6) that there is only one way in which differentials can act. Namely,

$$p^{i-1}y \text{ and } \{y_i, y_{i+1}, \ldots, y_m\} \text{ survive to } \hat{B}_i \text{ and we have } \hat{d}_i(p^{i-1}y) = y_i \text{ for } 1 \leqslant i \leqslant m. \tag{9:13}$$

In particular, since y_i survives nontrivially to $\hat{B}_i$ it follows from 7:5 that $v_1^{i-1} z_i \neq 0$.

(B) *The case* $\tilde{H}^*(Y; \mathbf{Z}/p) = \Sigma^s L(\infty)$ *for some* $s \geqslant 0$. The argument in part (A) extends in an obvious way to show that $BP\langle 1\rangle^*(Y)$ contains an infinite collection of elements $\{z_1, z_2, \ldots\}$ satisfying 9:3 and 9:4 (except of course for the relation $pz_m = 0$).

(C) *The case free* $E(Q_0, Q_1)$ *modules in* $H^*(Y; \mathbf{Z}/p)$. Spectral sequence arguments as in part (A) show that each free $E(Q_0, Q_1)$ summand of $H^*(Y; \mathbf{Z}/p)$ produces two $\mathbf{Z}/p$ summands in $H^*(Y)$ and one $BP\langle 1\rangle^*(\mathrm{pt})/(p, v_1)$ $(= \mathbf{Z}/p)$ summand in $BP\langle 1\rangle^*(Y)$. In particular, suppose that $\tilde{H}^*(Y; \mathbf{Z}/p) = \Sigma^s L(t) \oplus F$ where F is a free $E(Q_0, Q_1)$ module. Then when we calculate the various Bockstein spectral sequences as in part (A) the action of the differentials produced by the free summands $E(Q_0, Q_1)$ is completely independent of the action produced by the summand $L(t)$. This shows that the results obtained in (A) and (B) still hold if $H^*(Y; \mathbf{Z}/p) = \Sigma^s L(t) \oplus F$ where F is a free $E(Q_0, Q_1)$ module.

(D) *Proof of Theorem* 1:5. To calculate $BP\langle 1\rangle^*(\prod_{i=1}^m X)$ we can always replace $\prod_{i=1}^m X$ by its stable homotopy type. When we stabilize $\prod_{i=1}^m X$ we obtain the direct summand $X(m) = X \wedge \cdots \wedge X$ (m copies). For convenience we will only look at $BP\langle 1\rangle^*(X(m))$. By 8:2 and 9:2, $H^*(X(m); \mathbf{Z}/p)$ contains the summand $\Sigma^{3m}L(m)$. Let $Y = X^{2p+2}$, the $2p+2$ skeleton of X. By 9:2, $\tilde{H}^*(Y; \mathbf{Z}/p) \cong \Sigma^3 L(1)$. Let $Y(m) = Y \wedge \cdots \wedge Y$ (m copies). Then, by 8:2, $\tilde{H}^*(Y(m); \mathbf{Z}/p) \cong \Sigma^{3m}L(m) \oplus F$. Let f: $Y(m) \to X(m)$ be the natural inclusion. Then

the map f: $Y(m) \to X(m)$ induces an isomorphism α: $\Sigma^{3m}L(m) \subset H^*(X(m); \mathbf{Z}/p) \to H^*(Y(m); \mathbf{Z}/p) \to \Sigma^{3m}L(m)$. (9:14)

Since we are working stably we can replace $X(m)$ and $Y(m)$ by their suspension spectra and pass to the stable category of CW spectra. Consider the Eilenberg-Mac Lane spectrum $K = K(\mathbf{Z}_{(p)}, 0)$. By using some ideas of Frank Peterson it follows that

$$H^*(K; \mathbf{Z}/p) \cong L(\infty) \oplus F \quad \text{where } F \text{ is a free } E(Q_0, Q_1) \text{ module.} \qquad (9:15)$$

PROOF. We can identify $H^*(K; \mathbf{Z}/p)$ as a Steenrod module with $B^* = A^*(p)/A^*(p)Q_0$. When we dualize we have $B_* \subset A_*(p) = E(\tau_0, \tau_1, \ldots) \otimes \mathbf{Z}/p[\xi, \xi_2, \ldots]$ (the dual of the Steenrod algebra). Applying the canonical antiautomorphism χ we have $\chi(B_*) \subset A_*(p)$ where $\chi(B_*)$ denotes the image of B_* under χ. It turns out that the simplest approach is to describe the action of $E(Q_0, Q_1)$ on $\chi(B_*)$. This is mainly because

$$\chi(B_*) = E(\tau_1, \tau_2, \ldots) \otimes \mathbf{Z}/p[\xi_1, \xi_2, \ldots]. \qquad (9:15:1)$$

To prove 9:15:1 we will use R and L to denote right and left actions of $A^*(p)$. We have the following exact sequences, each derived from the previous one:

$$A^*(p) \overset{R(Q_0)}{\to} A^*(p) \to B^* \to 0,$$

$$A^*(p) \overset{L(Q_0)}{\leftarrow} A_*(p) \leftarrow B_* \leftarrow 0,$$

$$A_*(p) \overset{R(Q_0)}{\leftarrow} A_*(p) \leftarrow \chi(B_*) \leftarrow 0.$$

The right action of Q_0 is determined by the rules: $\tau_0 Q_0 = 1$, $\tau_s Q_0 = \xi_s Q_0 = 0$ for $s \geqslant 1$. (This can be read off the coalgebra structure of $A_*(p)$. See Theorem 3 of [**22**].) We can now easily deduce 9:15:1.

We are interested in the left action of $E(Q_0, Q_1)$ on $\chi(B_*)$. This action is determined by the rules: $Q_0\tau_s = \xi_s$ and $Q_1\tau_s = \xi_{s-1}^p$ ($\xi_0 = 1$) for $s > 1$ (again consider Theorem 3 of [**22**]). Thus $H(\chi(B_*); Q_0) = \mathbf{Z}/p$ is generated by 1 while $H(\chi(B_*); Q_1) = \{0\}$. The inclusion γ: $E(\tau_1) \otimes \mathbf{Z}/p[\xi_1] \to \chi(B_*)$ is a map of left $E(Q_0, Q_1)$ modules and induces an isomorphism in Q_0 and Q_1 homology. It follows from 16:3 of Part III of [**2**] that $E(\tau_1) \otimes \mathbf{Z}/p[\xi_1]$ and $\chi(B_*)$ are stably isomorphic. Since γ is injective it follows that $\chi(B_*) = E(\tau_1) \otimes \mathbf{Z}/p[\xi_1] \oplus F$ where F is a free $E(Q_0, Q_1)$ module. Dualizing we obtain 9:15. □

It follows from 9:5 that $\Sigma^{3m}L(m) \subset H^*(X(m); \mathbf{Z}/p)$ produces a summand $\mathbf{Z}_{(p)} \subset H^{3m}(X(m))$. Let $g\colon X(m) \to \Sigma^{3m}K(\mathbf{Z}_{(p)}, 0)$ be a map classifying a generator of this summand.

The map $g\colon X(m) \to \Sigma^{3m}K(\mathbf{Z}_{(p)}, 0)$ induces a surjective map $\beta\colon\ \Sigma^{3m}L(\infty) \subset H^*(\Sigma^{3m}K;\ \mathbf{Z}/p) \to H^*(X(m);\ \mathbf{Z}/p) \to \Sigma^{3m}L(m)$. (9:16)

PROOF. The map β is a map of $E(Q_0, Q_1)$ modules. By the definition of g, β induces an isomorphism in dimension $3m$. It is then easy to deduce from the $E(Q_0, Q_1)$ structure of $L(\infty)$ and $L(m)$ that β must be surjective. □

We now prove Theorem 1:5. By 7:5 it suffices to take the Bockstein spectral sequence $\{\hat{B}_r\}$ for $n = 1$ and show that when applied to $X(m)$ we have $\hat{d}_m \neq 0$ in $\hat{B}_m$. If we take $\{\hat{B}_r\}$ for the space $Y(m)$ then it follows from parts (A) and (C) that we can find $x \in \hat{B}_m^{3m}$ where $\hat{d}_m(x) \neq 0$. Similarly if we take $\{\hat{B}_r\}$ for the space $\Sigma^{3m}K$ then it follows from part (B) and (C) that we can find $y \in \hat{B}_m^{3m}$ such that $\hat{d}_m(y) \neq 0$. Moreover, it follows from 9:14 and 9:16 that we can let $x = (gf)^*(y)$. It now follows that if we take $\{\hat{B}_r\}$ for the space $X(m)$ then $\hat{d}_m g^*(y) \neq 0$ in $\hat{B}_m$.

REMARK 9:17. A more detailed analysis would allow us to conclude that the entire situation described in 9:3 occurs in $BP\langle 1\rangle^*(X(m))$. In particular, there is higher p torsion occurring in $BP\langle 1\rangle^*(X(m))$ as well.

REFERENCES

1. J. F. Adams, *Lectures on generalized cohomology*, Lecture Notes in Math., vol. 99, Springer-Verlag, Berlin and New York, 1969.

2. ______, *Stable homotopy and generalized homotopy*, Univ. of Chicago Press, Chicago, Ill., 1974.

3. W. Browder, *Torsion in H-spaces*, Anm. of Math. (2) **74** (1961), 24–51.

4. ______, *On differential Hopf algebras*, Trans. Amer. Math. Soc. **107** (1963), 153–176.

5. ______, *On higher torsion in H-spaces*, Trans. Amer. Math. Soc. **108** (1963), 353–375.

6. W. Browder and E. Spanier, *H-spaces and duality*, Pacific J. Math. **12** (1962), 411–414.

7. A. Clark, *Homotopy commutativity and the Moore spectral sequence*, Pacific J. Math. **15** (1965), 419–426.

8. J. Harper, *H-spaces with torsion*, Mem. Amer. Math. Soc. No. 223 (1979).

9. M. Hazewinkel, *A universal formal group and complex cobordism*, Bull. Amer. Math. Soc. **81** (1975), 930–933.

10. L. Hodgkins, *On the K-theory of Lie groups*, Topology **6** (1966), 1–36.

11. D. C. Johnson and W. S. Wilson, *Projective dimension and Brown-Peterson homology*, Topology **12** (1973), 327–353.

12. ______, *BP operations and Morava's extraordinary K-theories*, Math. Z. **144** (1975), 55–75.

13. R. Kane, *Loop spaces without p torsion*, Pacific J. Math. **60** (1975), 189–201.

14. ______, *On torsion in homotopy associative H-spaces*, Illinois J. Math. **20** (1976), 476–485.

15. ______, *The BP homology of H-spaces*, Trans. Amer. Math. Soc. **241** (1978), 99–120.

16. ______, *BP homology and finite H-spaces*, Lecture Notes in Math., vol. 673, Springer-Verlag, Berlin and New York, 1978, pp. 93–105.

17. P. S. Landweber, *Kunneth formulas for bordism theories*, Trans. Amer. Math. Soc. **121** (1966), 242–256.

18. J. Lin, *Torsion in H-spaces*. I, Ann. of Math. (2) **103** (1976), 457–487.

19. ______, *Torsion in H-spaces*. II, Ann. of Math. (2) **107** (1978), 41–88.

20. J. Milnor, *Construction of universal bundles*. I, Ann. of Math. (2) **63** (1956), 272–284.

21. ______, *Construction of universal bundles*. II, Ann. of Math. (2) **63** (1956), 430–436.

22. ______, *The Steenrod algebra and its dual*, Ann. of Math. (2) **67** (1958), 150–171.

23. J. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Ann. of Math. (2) **81** (1965), 211–264.

24. T. Petrie, *The weakly complex bordism of Lie groups*, Ann. of Math. (2) **88** (1968), 370–402.

25. D. G. Quillen, *On the formal group laws of unoriented and complex cobordism theory*, Bull. Amer. Math. Soc. **75** (1969), 1293–1298.

26. U. Würgler, *On products in a family of cohomology theories associated to the invariant prime ideals of* $\pi_*(BP)$, Comment. Math. Helv. **52** (1977), 239–260.

27. N. Yagita, *On mod odd prime Brown-Peterson cohomology groups of exceptional Lie groups* (to appear).

28. ______, *The* BP^**-module structure of* $BP^*(E_8)$ *for* $p = 3$ (to appear).

29. R. Zahler, *The Adams-Novikov spectral sequence for the spheres*, Ann. of Math. (2) **96** (1972), 480–504.

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